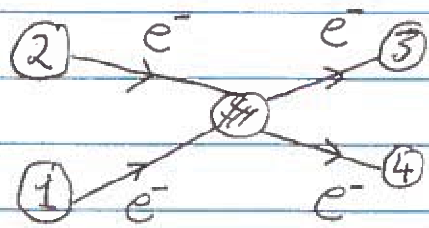
Physics 8802.01 Solutions to Set #9

1.



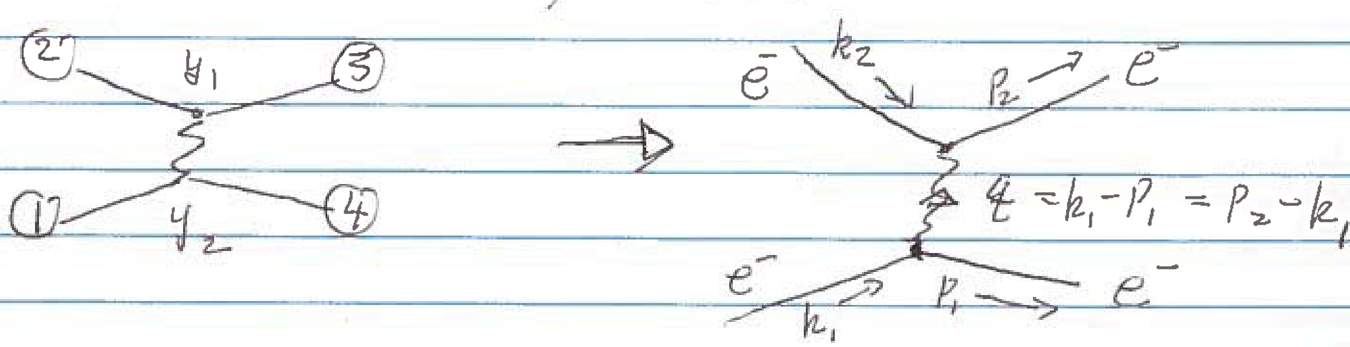
Electrons created at ① & ② $\Rightarrow \bar{\psi}(1)\,\bar{\psi}(2)$

Electrons annihilated at ③ & ④ $\Rightarrow \psi(3)\,\psi(4)$

There are two distinct diagrams (not including $y_1 \rightleftarrows y_2$)

Ⓐ $$\bar{\psi}(1)\,\bar{\psi}(2)\,\psi(3)\,\psi(4)\,\left[(\bar{\psi}\gamma^{\nu}\psi)A_{\nu}\right]_{y_1}\left[(\bar{\psi}\gamma^{5}\psi)A_{5}\right]_{y_2}$$

Contract in order ② (+), then ① (−), then ④ (−) then ③ (+) $\Rightarrow$ total sign $\equiv \boxed{+1}$

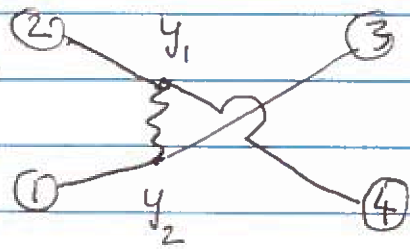


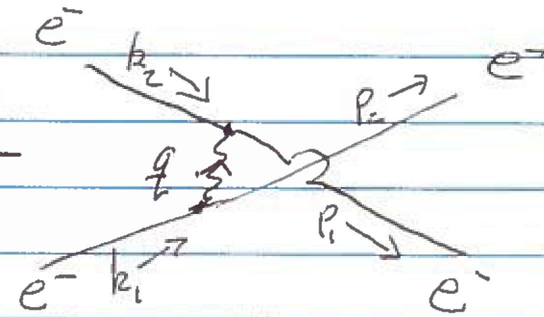
(B)

$$\bar\psi(1)\,\bar\psi(2)\,\psi(3)\,\psi(4)\,\left[(\bar\psi\gamma^\nu\psi)A_\nu\right]_{y_1}\left[(\bar\psi\gamma^\rho\psi)A_\rho\right]_{y_2}$$

Contract in order ② (+), then ① (−), then ④ (+), then ③ (+)

$\Rightarrow$ total sign $= \boxed{-1}$

$-$ 

$\Rightarrow$ $-$ 

$$q = k_1 - p_2 = p_1 - k_2$$

$$\boxed{\begin{aligned} i\mathcal{M}_{fi} = (-ie)^2 \Big\{ & \left(\bar u(p_1,s_1')\,\gamma^\nu\, u(k_1,s_1)\right)\left(\bar u(p_2,s_2')\,\gamma_\nu\, u(k_2,s_2)\right) * \frac{-i}{(k_1-p_1)^2+i\varepsilon} \\ & - \left(\bar u(p_1,s_1')\,\gamma^\rho\, u(k_2,s_2)\right)\left(\bar u(p_2,s_2')\,\gamma_\rho\, u(k_1,s_1)\right)\frac{-i}{(k_1-p_2)^2+i\varepsilon}\Big\} \end{aligned}}$$

2.

$$d\sigma = \frac{1}{4}\sum_{spins} |M_{fi}|^2 \frac{1}{4\sqrt{(k_1\cdot k_2)^2 - m_1^2 m_2^2}} (2\pi)^4 \delta^{(4)}(k_1+k_2-p_1-p_2) *$$

$$* \left(\prod_{j=1,2} \frac{d^3 p_j}{(2\pi)^3\, 2E_{p_j}} \right)$$

$$\overline{|M_{fi}|^2} \equiv \frac{1}{4}\sum_{spins} |M_{fi}|^2 =$$

$$= \frac{e^4}{4} \sum_{s_1, s_2, s_1', s_2'} \Big\{ \Big[\frac{1}{(k_1-p_1)^4} \left| \left(\bar{u}(p_1,s_1')\gamma^\nu u(k_1,s_1)\right)\left(\bar{u}(p_2,s_2')\gamma_\nu u(k_2,s_2)\right)\right|^2 \Big]$$

$$+ \Big[\quad (p_1, s_1') \rightleftarrows (p_2, s_2') \quad \Big]$$

$$- \Big[\frac{1}{(k_1-p_1)^2 (k_1-p_2)^2} \Big(\left(\bar{u}(p_1,s_1')\gamma^\nu u(k_1,s_1)\right)\left(\bar{u}(p_2,s_2')\gamma_\nu u(k_2,s_2)\right)$$

$$* \left(\bar{u}(k_2,s_2)\gamma^\rho u(p_1,s_1')\right)\left(\bar{u}(k_1,s_1)\gamma_\rho u(p_2,s_2')\right)\Big) + h.c. \Big] \Big\}$$

The first 2 terms inside $\{\ \}$ are similar to what we had for $e^+e^- \to \mu^+\mu^-$. Using

$$\sum_s u^\alpha(p,s)\,\bar{u}^\beta(p,s) = (\not{p} + m_e)_{\alpha\beta} \quad \text{and ignoring } m_e$$

$\underline{\text{First 2 terms}}$

$$= \frac{e^4}{4}\Big\{ \frac{1}{(k_1-p_1)^4} \, tr\{\not{p}_1 \gamma^\nu \not{k}_1 \gamma^\rho\} \cdot tr\{\not{p}_2 \gamma_\nu \not{k}_2 \gamma_\rho\}$$

$$+ \frac{1}{(k_1-p_2)^4} \, tr\{\not{p}_2 \gamma^\nu \not{k}_1 \gamma^\rho\} \cdot tr\{\not{p}_1 \gamma_\nu \not{k}_2 \gamma_\rho\} \Big\}$$

Similarly the third (cross) term gives

$$\underline{\text{cross term}} = \frac{e^4}{4}\frac{-1}{(k_1-p_1)^2(k_1-p_2)^2}\left[\mathrm{tr}\{\not{p}_1 \gamma^\nu \not{k}_1 \gamma_\rho \not{p}_2 \gamma_\nu \not{k}_2 \gamma^\rho\} + h.c. \right]$$

Use

$$\mathrm{tr}\{\not{p}_1 \gamma^\nu \not{k}_1 \gamma^\rho\} = 4\left[p_1^\nu k_1^\rho + p_1^\rho k_1^\nu - (p_1\cdot k_1)\, g^{\nu\rho} \right]$$

$$\mathrm{tr}\{\not{p}_2 \gamma_\nu \not{k}_2 \gamma_\rho\} = 4\left[p_{2\nu} k_{2\rho} + p_{2\rho} k_{2\nu} - (p_2\cdot k_2)\, g_{\nu\rho} \right]$$

$\Rightarrow$

$$\boxed{\mathrm{tr}\{\not{p}_1 \gamma^\nu \not{k}_1 \gamma^\rho\}\cdot \mathrm{tr}\{\not{p}_2 \gamma_\nu \not{k}_2 \gamma_\rho\} = 32\left[(p_1\cdot p_2)(k_1\cdot k_2) + (p_1\cdot k_2)(k_1\cdot p_2) \right]}$$

Also from $p_1 \rightleftarrows p_2$

$$\boxed{\mathrm{tr}\{\not{p}_2 \gamma^\nu \not{k}_1 \gamma^\rho\}\cdot \mathrm{tr}\{\not{p}_1 \gamma_\nu \not{k}_2 \gamma_\rho\} = 32\left[(p_1\cdot p_2)(k_1\cdot k_2) + (p_2\cdot k_2)(k_1\cdot p_1) \right]}$$

For the "cross term" use H. & M. eq (6.24) repeatedly

$$\mathrm{tr}\{\not{p}_1 \gamma^\nu \not{k}_1 \gamma_\rho \not{p}_2 \gamma_\nu \not{k}_2 \gamma^\rho\} =$$

$$= -2\,\mathrm{tr}\{\not{p}_1 \not{p}_2 \gamma_\rho \not{k}_1 \not{k}_2 \gamma^\rho\} = -8\,(k_1\cdot k_2)\,\mathrm{tr}\{\not{p}_1 \not{p}_2\}$$

$$= \boxed{-32\left[(k_1\cdot k_2)(p_1\cdot p_2)\right]}$$

$$\boxed{\begin{aligned} |\mathcal{M}_{fi}|^2 = e^4 \Big\{ & \frac{8}{(k_1-p_1)^4}\left[(p_1\cdot p_2)(k_1\cdot k_2) + (p_1\cdot k_2)(k_1\cdot p_2) \right] \\ & + \frac{8}{(k_1-p_2)^4}\left[(p_1\cdot p_2)(k_1\cdot k_2) + (p_2\cdot k_2)(k_1\cdot p_1) \right] \\ & + \frac{16}{(k_1-p_1)^2(k_1-p_2)^2}\left[(p_1\cdot p_2)(k_1\cdot k_2) \right] \Big\} \end{aligned}}$$

In CM one has:

$$k_1 : (E, \vec{k}) \quad , \quad k_2 : (E, -\vec{k})$$
$$p_1 : (E', \vec{p}) \quad , \quad p_2 : (E', -\vec{p})$$

$$(k_1 \cdot k_2) = E^2 + \vec{k}^2 \approx 2E^2$$
$$(p_1 \cdot p_2) \approx 2E'^2$$
$$(k_1 \cdot p_1) = EE'(1 - \cos\theta) = 2EE' \sin^2\frac{\theta}{2} = (k_2 \cdot p_2)$$
$$(k_1 \cdot p_2) = EE'(1 + \cos\theta) = 2EE' \cos^2\frac{\theta}{2} = (k_2 \cdot p_1)$$

$$(k_1 - p_1)^2 = -2(k_1 \cdot p_1) = -4EE' \sin^2\frac{\theta}{2}$$
$$(k_1 - p_2)^2 = -2(k_1 \cdot p_2) = -4EE' \cos^2\frac{\theta}{2}$$

One also has

$$\int d^3p_1 \int d^3p_2 \, \delta^{(4)}(k_1 + k_2 - p_1 - p_2) \longrightarrow \int d^3p \, \delta(2E - 2E')$$

$$\sigma = \int \frac{d^3p}{(2\pi)^2} \frac{1}{4(2E^2)} \frac{1}{(2E')^2} \delta(2E - 2E') \overline{|\mathcal{M}_{fi}|^2}$$

$$= \int \frac{d\Omega}{64E^2} \frac{1}{(2\pi)^2} \overline{|\mathcal{M}_{fi}|^2} \Big|_{E'=E}$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{64E^2} \frac{1}{(2\pi)^2} |\mathcal{M}_{fi}|^2 \Big|_{E'=E}}$$

$$\overline{|\mathcal{M}_{fi}|^2}\Big|_{E'=E} = e^4 \left\{ \frac{8}{16E^4 \sin^4\frac{\theta}{2}} \left[(2E^2)^2 + (2E^2)^2 \cos^4\frac{\theta}{2} \right] \right.$$

$$+ \frac{8}{16E^4 \cos^4\frac{\theta}{2}} \left[(2E^2)^2 + (2E^2)^2 \sin^4\frac{\theta}{2} \right]$$

$$\left. + \frac{16}{16E^4 \cos^2\frac{\theta}{2} \cdot \sin^2\frac{\theta}{2}} (2E^2)^2 \right\}$$

Let $\alpha \equiv \frac{e^2}{4\pi}$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{8E^2} \left\{ \frac{(1+\cos^4\frac{\theta}{2})}{\sin^4\frac{\theta}{2}} + \frac{(1+\sin^4\frac{\theta}{2})}{\cos^4\frac{\theta}{2}} + \frac{2}{\sin^2\frac{\theta}{2} \cdot \cos^2\frac{\theta}{2}} \right\}}$$