P8802.01 Solutions to Set #8

1. $[S_F(x-y)]_{\alpha\beta} \equiv \langle 0 | T\{\psi_\alpha(x)\, \bar\psi_\beta(y)\} | 0\rangle =$

$$= \theta(x^0-y^0)\langle 0|\psi_\alpha(x)\bar\psi_\beta(y)|0\rangle \quad \theta(y^0-x^0)\langle 0|\bar\psi_\beta(y)\psi_\alpha(x)|0\rangle$$

$$\langle 0|\psi_\alpha(x)\bar\psi_\beta(y)|0\rangle = \sum_{s s'} \int \mathcal{D}_p \int \mathcal{D}_{p'}\, u_\alpha(p,s)\, \bar u_\beta(p',s')\, e^{-ip\cdot x}\, e^{ip'\cdot y}$$

$$* \langle 0|\underbrace{\hat b_{p s}\, \hat b^\dagger_{p',s'}}_{\text{replace by anticommutator}}|0\rangle$$

$$\langle 0|\psi_\alpha(x)\bar\psi_\beta(y)|0\rangle = $$
$$= \sum_s \int \mathcal{D}_p\, u_\alpha(p,s)\, \bar u_\beta(p,s)\, e^{-ip\cdot(x-y)}$$

Similarly: $\langle 0|\bar\psi_\beta(y)\psi_\alpha(x)|0\rangle = \sum_s \int \mathcal{D}_p\, v_\alpha(p,s)\, \bar v_\beta(p,s)\, e^{ip\cdot(x-y)}$

So, $$[S_F(x-y)]_{\alpha\beta} = \int \mathcal{D}_p \sum_s \Big\{ u_\alpha(p,s)\bar u_\beta(p,s)\, e^{-ip\cdot(x-y)}\theta(x^0-y^0)$$
$$- v_\alpha(p,s)\, \bar v_\beta(p,s)\, e^{ip\cdot(x-y)}\, \theta(y^0-x^0)\Big\}$$

$$= \int \mathcal{D}_p \Big\{ (\not p + m)_{\alpha\beta}\, e^{-ip\cdot(x-y)}\, \theta(x^0-y^0) - (\not p - m)_{\alpha\beta}\, e^{ip\cdot(x-y)}\theta(y^0-x^0)\Big\}$$

Use
$$\begin{cases} \theta(x^0-y^0)\, e^{-iE_p(x^0-y^0)} = i\displaystyle\int \frac{dp_0}{2\pi}\, \frac{e^{-ip_0(x^0-y^0)}}{p_0-E_p+i\varepsilon} \\ \theta(y^0-x^0)\, e^{-iE_p(y^0-x^0)} = i\displaystyle\int \frac{dp_0}{2\pi}\, \frac{e^{-ip_0(y^0-x^0)}}{p_0-E_p+i\varepsilon} \end{cases}$$

$$[S_F(x-y)]_{\alpha\beta} = i\int \frac{d^4p}{(2\pi)^4\, 2E_p}\Big\{(\not p+m)_{\alpha\beta}\, \frac{e^{ip\cdot(x-y)}}{p_0-E_p+i\varepsilon} - (\not p-m)_{\alpha\beta}\, \frac{e^{ip\cdot(x-y)}}{p_0-E_p+i\varepsilon}\Big\}$$

with $p\cdot(x-y) = p_0(x^0-y^0) - \vec p\cdot(\vec x-\vec y)$

Let $p_\mu \to -p_\mu$ in 2^{nd} term

$$\left[S_F(x-y)\right]_{\alpha\beta} = i\int \frac{d^4p}{(2\pi)^4\, 2E_p} (\not p + m)_{\alpha\beta}\, e^{-ip\cdot(x-y)}$$

$$* \left[\frac{1}{p_0 - E_p + i\varepsilon} + \frac{1}{-p_0 - E_p + i\varepsilon}\right]$$

$$\left[\frac{1}{p_0 - E_p + i\varepsilon} + \frac{1}{-p_0 - E_p + i\varepsilon}\right] = \frac{2(E_p - i\varepsilon)}{p_0^2 - (E_p - i\varepsilon)^2} \longrightarrow \frac{2E_p}{p^2 - m^2 + i\tilde{\varepsilon}}$$

$$p_0^2 - (E_p - i\varepsilon)^2 = p_0^2 - (\vec{p}^{\,2} + m^2) + \underbrace{2i\varepsilon E_p}_{\equiv i\tilde{\varepsilon}} + \mathcal{O}(\varepsilon^2) = p^2 - m^2 + i\tilde{\varepsilon}$$

Rename $\tilde{\varepsilon} \to \varepsilon$

$$\boxed{S_F(x-y) = i\int \frac{d^4p}{(2\pi)^4} \frac{(\not p + m)}{p^2 - m^2 + i\varepsilon} e^{-ip\cdot(x-y)}}$$

One sees that for

$\underline{x^0 > y^0}$: $\langle 0 | \hat{b}\, \hat{b}^\dagger | 0 \rangle$ operative
particle created at y
annih. at x

$x \longleftarrow y$
particle

$\underline{y^0 > x^0}$ $\langle 0 | \hat{d}\, \hat{d}^\dagger | 0 \rangle$ operative
antiparticle created at x
annih. at y

$x \longrightarrow y$
antiparticle

Either way, creation occurs before annihilation.
In both cases "fermion number" flows from y to x $\quad x \longleftarrow y$

2/ Regardless of various normalization conventions one must have

$$\boxed{\{\hat{\psi}_\alpha(\vec{x},t)\,,\,\hat{\Pi}_\beta(\vec{y},t)\}_{ETC} = i\,\delta_{\alpha\beta}\,\delta^{(3)}(\vec{x}-\vec{y})}$$

$$\Pi_\beta = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_\beta)} = i\,(\bar{\psi}\gamma_0)_\beta = i\,\psi^\dagger_\beta$$

$$\hat{\psi}(x) = \sum_s \int \frac{d^3p}{(2\pi)^3} N_p \{\hat{b}_{p,s}\, u\, e^{-ip\cdot x} + \hat{d}^\dagger_{ps}\, v\, e^{ip\cdot x}$$

$$\{\hat{\psi}_\alpha(\vec{x},t)\,,\,\hat{\psi}^\dagger_\beta(\vec{y},t)\}_{ETC} =$$

$$= \sum_{s,s'} \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} N_p N_{p'} (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}')\, \delta_{s,s'}\, C_p$$
$$[u_\alpha u^\dagger_\beta\, e^{i\vec{p}\cdot(\vec{x}-\vec{y})} + v_\alpha v^\dagger_\beta\, e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}]$$

$$= \sum_s \int \frac{d^3p}{(2\pi)^3} N_p^2\, C_p \Big[\big(u_\alpha(p,s)\, u^\dagger_\beta(p,s)\big)\, e^{i\vec{p}\cdot(\vec{x}-\vec{y})}$$
$$+ \big(v_\alpha(p,s)\, v^\dagger_\beta(p,s)\big)\, e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\Big]$$

Use $\sum_s \big(u_\alpha(p,s)\, u^\dagger_\beta(p,s)\big) = \sum_s u_\alpha (\bar{u}\gamma^0)_\beta =$

$$= \sum_s \sum_{\rho=1}^{4} u_\alpha \bar{u}_\rho\, \gamma^0_{\rho\beta} = \sum_\rho R\,(\not p + m)_{\alpha\rho}\, \gamma^0_{\rho\beta}$$

$$= R\left[(\not p + m)\gamma^0\right]_{\alpha\beta}$$

Similarly $\sum_s \big(v_\alpha(p,s)\, v^\dagger_\beta(p,s)\big) = R\left[(\not p - m)\gamma^0\right]_{\alpha\beta}$

$$\{\hat{\psi}_\alpha(x), \hat{\psi}^\dagger_\beta(y)\}_{ETC} =$$

$$= \int \frac{d^3p}{(2\pi)^3} N_p^2 C_p R \left\{ [(\not{p}+m)\gamma^0]_{\alpha\beta} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} + [(\not{p}-m)\gamma^0]_{\alpha\beta} e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} \right\}$$

$$(\not{p}+m)\gamma^0 = [E_p - \vec{p}\cdot\vec{\gamma}\gamma^0 + m\gamma^0]$$

$$(\not{p}-m)\gamma^0 = [E_p - \vec{p}\cdot\vec{\gamma}\gamma^0 - m\gamma^0] \xrightarrow[\substack{\vec{p}\to-\vec{p} \\ \text{inside integral}}]{} [E_p + \vec{p}\cdot\vec{\gamma}\gamma^0 - m\gamma^0]$$

$$\{\hat{\psi}_\alpha(\vec{x},t), \hat{\psi}^\dagger_\beta(\vec{y},t)\}_{ETC} = \delta_{\alpha\beta} \int \frac{d^3p}{(2\pi)^3} N_p^2 C_p \cdot R\, 2E_p\, e^{i\vec{p}\cdot(\vec{x}-\vec{y})}$$

Want this to equal $\delta_{\alpha\beta}\, \delta^{(3)}(\vec{x}-\vec{y}) = \delta_{\alpha\beta} \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})}$

$$\Rightarrow \boxed{N_p^2 C_p R\, 2E_p = 1}$$

3.

Let $t_1 > t_2 > t_3$, then

$$T\{\varphi_1, \varphi_2, \varphi_3\} = \varphi_1\varphi_2\varphi_3 = (\varphi_1^+ + \varphi_1^-)(\varphi_2^+ + \varphi_2^-)(\varphi_3^+ + \varphi_3^-)$$

$$= (\varphi_1^+ + \varphi_1^-)\left[\varphi_2^+\varphi_3^+ + \varphi_2^-\varphi_3^+ + \varphi_2^+\varphi_3^- + \varphi_2^-\varphi_3^-\right]$$

$$= \varphi_1^-\left(\varphi_2^+\varphi_3^+ + \varphi_2^-\varphi_3^+ + \varphi_3^-\varphi_2^+ + [\varphi_2^+, \varphi_3^-] + \varphi_2^-\varphi_3^-\right)$$

$$+ \varphi_1^+\left(\quad - \text{''} - \quad - \text{''} - \quad - \text{''} - \quad - \text{''} - \quad \right) \qquad (1)$$

The commutator $[\varphi_2^+, \varphi_3^-] =$

$$= \int \mathcal{D}p_2 \int \mathcal{D}p_3 \; e^{-ip_2\cdot x_2}\, e^{ip_3 x_3}\, [\hat{a}_{p_2}, \hat{a}^+_{p_3}]$$

$$= \int \mathcal{D}_p\, e^{-ip\cdot(x_2 - x_3)} \equiv D(x_2 - x_3) \qquad (2)$$

= just a c-number (not an operator)

$\Rightarrow$ 1^{st} line in (1) is already normal ordered and we can focus on 2^{nd} line

$$2^{nd}\text{ line} = \varphi_1^+\varphi_2^+\varphi_3^+ + \varphi_2^-\varphi_1^+\varphi_3^+ + [\varphi_1^+, \varphi_2^-]\varphi_3^+$$

$$+ \varphi_3^-\varphi_1^+\varphi_2^+ + [\varphi_1^+, \varphi_3^-]\varphi_2^+ + \varphi_1^+[\varphi_2^+, \varphi_3^-]$$

$$+ \varphi_2^-\varphi_1^+\varphi_3^- + [\varphi_1^+, \varphi_2^-]\varphi_3^- \qquad (3)$$

$$\equiv \varphi_2^-\varphi_3^-\varphi_1^+ + \varphi_2^-[\varphi_1^+, \varphi_3^-]$$

$$(1) = \left[8 \text{ normal ordered terms with } 3\,\varphi^{\pm} \right] + (\varphi_1^+ + \varphi_1^-)\, D(x_2 - x_3) + (\varphi_2^+ + \varphi_2^-)\, D(x_1 - x_3) + (\varphi_3^+ + \varphi_3^-)\, D(x_1 - x_2)$$

Note that a single $\varphi_i = :\varphi_i:$

Also for $t_i > t_j$ one has $D(x_i - x_j) = D_F(x_i - x_j)$

$$T\{\varphi_1 \varphi_2 \varphi_3\}\Big|_{t_1 > t_2 > t_3} = :\varphi_1 \varphi_2 \varphi_3: + :\left(\varphi_1 D_F(x_2 - x_3) + \varphi_2 D_F(x_1 - x_3) + \varphi_3 D_F(x_1 - x_2) \right): \qquad (4)$$

For other time orderings $t_i > t_j > t_k$ one has instead of (1) as starting point

$$T\{\varphi_1 \varphi_2 \varphi_3\} = \varphi_i \varphi_j \varphi_k = \cdots\cdots$$

Everything goes through as before and using again $D(x_i - x_j) = D_F(x_i - x_j)$ for $t_i > t_j$ one ends up again with the RHS of (4).

Hence in general.

$$\boxed{T\{\varphi_1 \varphi_2 \varphi_3\} = :\varphi_1 \varphi_2 \varphi_3: + :\left(\varphi_1 \overbrace{\varphi_2 \varphi_3} + \overbrace{\varphi_1 \varphi_2 \varphi_3} + \varphi_1 \overbrace{\varphi_2 \varphi_3} \right):}$$

where $\overbrace{\varphi_i \varphi_j} \equiv D_F(x_i - x_j)$ (5)