P8802.01 Set #7 Solutions

1.(a)

$$\begin{cases} \mathcal{L} = \partial_\mu \varphi^* \, \partial^\mu \varphi - m^2 \varphi^* \varphi \\ \Pi = \dfrac{\partial \mathcal{L}}{\partial(\partial_0 \varphi)} = \partial^0 \varphi^* \end{cases}$$

Going to quantum fields

$$\begin{cases} \hat{\varphi} = \int \mathcal{D}_p \left[\hat{a}_p e^{-ip\cdot x} + \hat{b}_p^\dagger e^{ip\cdot x} \right] \\ \hat{\Pi} = \partial^0 \hat{\varphi}^\dagger = \int \mathcal{D}_p \, (iE_p) \left[\hat{a}_p^\dagger e^{ip\cdot x} - \hat{b}_p e^{-ip\cdot x} \right] \end{cases}$$

Let $\boxed{[\hat{a}_p, \hat{a}_{p'}^\dagger] = (2\pi)^3 \, 2E_p \, \delta^{(3)}(\vec{p} - \vec{p}\,') = [\hat{b}_p, \hat{b}_{p'}^\dagger]}$ (1)

and all other commutators $= \phi$

Then

$$[\hat{\varphi}(\vec{x},t), \hat{\Pi}(\vec{y},t)]_{ETC} =$$

$$= \int \mathcal{D}_p \int \mathcal{D}_{p'} \Big\{ [\hat{a}_p, \hat{a}_{p'}^\dagger] \, e^{-ip\cdot x} e^{ip'\cdot y} \, (iE_{p'})$$

$$- [\hat{b}_p^\dagger, \hat{b}_{p'}] \, e^{ip\cdot x} e^{-ip'\cdot y} \, (iE_{p'}) \Big\} =$$

$$= \int \mathcal{D}_p \, (iE_p) \, 2 \, e^{i\vec{p}\cdot(\vec{x}-\vec{y})} = i \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})}$$

$$= i \, \delta^{(3)}(\vec{x} - \vec{y})$$

So the commutators (1) are consistent with the canonical quantization condition

$$\boxed{[\hat{\varphi}(\vec{x},t), \hat{\Pi}(\vec{y},t)]_{ETC} = i \, \delta^{(3)}(\vec{x} - \vec{y})} \quad (2)$$

1. (b)

$$\hat{H} = \int T^{00}\, d^3x = \int \left[\partial^0 \hat{\varphi}^\dagger\, \partial_0 \hat{\varphi} + \vec{\nabla}\hat{\varphi}^\dagger \cdot \vec{\nabla}\hat{\varphi} + m^2 \hat{\varphi}^\dagger \hat{\varphi} \right] d^3x$$

$$= \int d^3x \int \mathcal{D}_p \int \mathcal{D}_{p'} \Big\{ E_p E_{p'} \left[(\hat{a}_p^\dagger e^{ip\cdot x} - \hat{b}_p e^{-ip\cdot x})(\hat{a}_{p'} e^{-ip'\cdot x} - \hat{b}_{p'}^\dagger e^{ip'\cdot x}) \right]$$

$$+ \vec{p}\cdot\vec{p}\,' \left[(\hat{a}_p^\dagger e^{ip\cdot x} - \hat{b}_p e^{-ip\cdot x})(\hat{a}_{p'} e^{-ip'\cdot x} - \hat{b}_{p'}^\dagger e^{ip'\cdot x}) \right]$$

$$+ m^2 \left[(\text{-''-} + \text{-''-})(\text{-''-} + \text{-''-}) \right] \Big\}$$

$$= \int \mathcal{D}_p \frac{1}{2E_p} \Big\{ E_p^2 \left[\hat{a}_p^\dagger \hat{a}_p - \hat{b}_p \hat{a}_{-p} e^{-2iE_p t} - \hat{a}_p^\dagger \hat{b}_{-p}^\dagger e^{2iE_p t} + \hat{b}_p \hat{b}_p^\dagger \right]$$

$$+ \vec{p}^{\,2} \left[\text{-''-} + \text{-''-} + \text{-''-} + \text{-''-} \right]$$

$$+ m^2 \left[\text{-''-} + \text{-''-} + \text{-''-} + \text{-''-} \right] \Big\}$$

$$= \int \mathcal{D}_p\, E_p \left[\hat{a}_p^\dagger \hat{a}_p + \hat{b}_p \hat{b}_p^\dagger \right]$$

$$\boxed{\hat{H} = \int \mathcal{D}_p\, E_p \left[\hat{a}_p^\dagger \hat{a}_p + \hat{b}_p^\dagger \hat{b}_p \right]} + \text{const.} \qquad (3)$$

Symmetry $\varphi \to e^{i\alpha}\varphi$ leads to conserved current

$$j^\mu = (\partial^\mu \varphi^*)\, i\varphi - i\varphi^* (\partial^\mu \varphi)$$

$$\Rightarrow \hat{Q} = \int d^3x\, \hat{j}^0 = i \int d^3x \left[(\partial^0 \hat{\varphi}^\dagger)\, \hat{\varphi} - \hat{\varphi}^\dagger (\partial^0 \hat{\varphi}) \right]$$

$$= i \int d^3x \int \mathcal{D}_p \int \mathcal{D}_{p'} \Big\{ (iE_p)(\hat{a}_p^\dagger e^{ip\cdot x} - \hat{b}_p e^{-ip\cdot x})(\hat{a}_{p'} e^{-ip'\cdot x} + b_{p'}^\dagger e^{ip'\cdot x})$$

$$- (\hat{a}_p^\dagger e^{ip\cdot x} + \hat{b}_p e^{-ip\cdot x})(-iE_{p'})(\hat{a}_{p'} e^{-ip'\cdot x} - b_{p'}^\dagger e^{ip'\cdot x}) \Big\}$$

$$= i \int \mathcal{D}_p \frac{1}{2E_p} (iE_p) \Big\{$$

$$[\hat{a}_p^\dagger \hat{a}_p - \hat{b}_p \hat{a}_{-p} e^{-2iE_pt} + \hat{a}_p^\dagger b_{-p}^\dagger e^{2iE_pt} - \hat{b}_p \hat{b}_p^\dagger]$$

$$+ [\text{-"-} + \text{-"-} \quad - \quad \text{-"-} \quad - \quad \text{-"-}] \Big\}$$

$$\boxed{\hat{Q} = -\int \mathcal{D}_p [\hat{a}_p^\dagger \hat{a}_p - \hat{b}_p^\dagger \hat{b}_p]} + \text{const.} \quad (4)$$

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1. (c) $\hat{H}|q\rangle = \left(\int \mathcal{D}_p E_p [\hat{a}_p^\dagger \hat{a}_p + \hat{b}_p^\dagger \hat{b}_p]\right) \hat{a}_q^\dagger |0\rangle$

Since $\hat{a}_p |0\rangle = 0 = \hat{b}_p |0\rangle$, the only surviving contribution is from the $\hat{a}_p^\dagger \hat{a}_p$ term with $\vec{p} = \vec{q}$

$$\hat{H}|q\rangle = \int \mathcal{D}_p E_p \hat{a}_p^\dagger \hat{a}_p \hat{a}_q^\dagger |0\rangle =$$

$$= \int \mathcal{D}_p E_p \hat{a}_p^\dagger \underbrace{[\hat{a}_p, \hat{a}_q^\dagger]}_{= (2\pi)^3 2E_q \delta^{(3)}(\vec{p} - \vec{q})} |0\rangle = E_q \hat{a}_q^\dagger |0\rangle = E_q |q\rangle$$

Similarly $\hat{H}|\bar{q}\rangle = E_q |\bar{q}\rangle$

$$\hat{Q}|q\rangle = -|q\rangle, \quad \hat{Q}|\bar{q}\rangle = +|\bar{q}\rangle$$

$\Rightarrow$ $\hat{a}_p^\dagger$ and $\hat{b}_p^\dagger$ create particles of same mass and opposite charge.
i.e. a particle and its antiparticle.

2.e)

$$\left[\begin{array}{l} \hat{\psi} = \sum_s \int \mathcal{D}_p \left[\hat{b}_{p,s}\, u(p,s)\, e^{-ip\cdot x} + \hat{d}^\dagger_{p,s}\, v(p,s)\, e^{ip\cdot x}\right] \\ \hat{\bar{\psi}} = \sum_s \int \mathcal{D}_p \left[\hat{b}^\dagger_{p,s}\, \bar{u}(p,s)\, e^{ip\cdot x} + \hat{d}_{p,s}\, \bar{v}(p,s)\, e^{-ip\cdot x}\right] \end{array}\right.$$

$$\hat{H} = \int d^3x\, \bar{\psi}\left[-i\vec{\gamma}\cdot\vec{\nabla} + m\right]\psi =$$

$$= \int d^3x \int \mathcal{D}_p \int \mathcal{D}_{p'} \sum_{s,s'} \Big\{ \left[\hat{b}^\dagger_{p,s}\, \bar{u}(p,s)\, e^{ip\cdot x} + \hat{d}_{p,s}\, \bar{v}(p,s)\, e^{-ip\cdot x}\right]$$

$$(-i\vec{\gamma}\cdot\vec{\nabla} + m)\left[\hat{b}_{p',s'}\, u(p',s')\, e^{-ip'\cdot x} + \hat{d}^\dagger_{p',s'}\, v(p',s')\, e^{ip'\cdot x}\right]\Big\}$$

Use fact that $u\, e^{-ip'\cdot x}$ and $v\, e^{ip'\cdot x}$ obey the Dirac Eq. and

$$-i\vec{\gamma}\cdot\vec{\nabla} + m = -\underbrace{(i\gamma^\mu \partial_\mu - m)}_{\substack{\text{gives zero when acting on} \\ u e^{-ip'\cdot x} \text{ etc.}}} + i\gamma^0 \partial_0$$

$$\hat{H} = \int d^3x \int \mathcal{D}_p \int \mathcal{D}_{p'} \sum_{s,s'} \Big\{ \big[\hat{b}^\dagger_{p,s}\, \bar{u}\, e^{ip x} + \hat{d}_{p,s}\, \bar{v}\, e^{-ip\cdot x} \big]$$

$$i\gamma^0 (-iE_{p'}) \big[\hat{b}_{p',s'}\, u(p',s')\, e^{-ip'\cdot x} - \hat{d}^\dagger_{p',s'}\, v\, e^{ip'\cdot x} \big] \Big\}$$

$$= \int \mathcal{D}_p \frac{1}{2E_p} E_p \sum_{s s'} \Big\{ \hat{b}^\dagger_{p,s}\, \hat{b}_{p,s'} \left(u^\dagger_{(p,s)} \cdot u_{(p,s')} \right)$$

$$+ \hat{d}_{p,s}\, \hat{b}_{-p,s'} \left(v^\dagger_{(p,s)} \cdot u_{(-p,s')} \right) e^{-2iE_p t}$$

$$- \hat{b}^\dagger_{p,s}\, \hat{d}^\dagger_{-p,s'} \left(u^\dagger_{(p,s)} \cdot v_{(-p,s')} \right) e^{2iE_p t} - \hat{d}_{p,s}\, \hat{d}^\dagger_{p,s'} \left(v^\dagger_{(p,s)} \cdot v_{(p,s')} \right) \Big\}$$

Use $u^\dagger_{(p,s)} \cdot u_{(p,s')} = 2E_p\, \delta_{s s'} = v^\dagger_{(p,s)} \cdot v_{(p,s')}$ etc.

$$\boxed{\hat{H} = \sum_s \int \mathcal{D}_p\, E_p \left[\hat{b}^\dagger_{p,s}\, \hat{b}_{p,s} + \hat{d}^\dagger_{p,s}\, \hat{d}_{p,s} \right]} + \text{const.} \tag{5}$$

Conserved Noether current $j^\mu = \bar{\psi}\gamma^\mu \psi$

$$\hat{Q} = \int d^3x\, j^0 = \int d^3x \left(\psi^\dagger \psi \right)$$

$$= \int d^3x \int \mathcal{D}_p \int \mathcal{D}_{p'} \Big\{ \big[b^\dagger_{p,s}\, u^\dagger e^{ip\cdot x} + \hat{d}_{p,s}\, v^\dagger\, e^{-ip\cdot x} \big]$$

$$\big[\hat{b}_{p',s'}\, u\, e^{-ip'\cdot x} + \hat{d}^\dagger_{p',s'}\, v\, e^{ip'\cdot x} \big] \Big\}$$

$$= \sum_s \int \mathcal{D}_p \left[\hat{b}^\dagger_{p,s}\, \hat{b}_{p,s} + \hat{d}_{p,s}\, \hat{d}^\dagger_{p,s} \right]$$

$$\boxed{\hat{Q} = \sum_s \int \mathcal{D}_p \left[\hat{b}^\dagger_{p,s}\, \hat{b}_{p,s} - \hat{d}^\dagger_{p,s}\, \hat{d}_{p,s} \right]} + \text{const.} \tag{6}$$

Again $\hat{b}^\dagger_{p,s}$ creates particle of mass m and charge $+1$ whereas $\hat{d}^\dagger_{p,s}$ creates its antiparticle with charge -1