Set #6 Solutions

1.(a) Following HW #3 Probs. 2 & 3 one can write

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi = i\bar{\psi}(P_L + P_R)\gamma^\mu(P_L + P_R)\partial_\mu\psi$$

Use $P_L\gamma^\mu P_L \equiv 0 \equiv P_R\gamma^\mu P_R$ and $\bar{\psi}_L = \bar{\psi}P_R$, $\bar{\psi}_R = \bar{\psi}P_L$

$$\Rightarrow \mathcal{L} = i\bar{\psi}P_L\gamma^\mu P_R\partial_\mu\psi + i\bar{\psi}P_R\gamma^\mu P_L\partial_\mu\psi$$

$$\boxed{\mathcal{L} = i\bar{\psi}_R\gamma^\mu\partial_\mu\psi_R + i\bar{\psi}_L\gamma^\mu\partial_\mu\psi_L}$$

Obviously $\mathcal{L}$ is invariant under separate phase rotations of ψ_L & ψ_R: $\psi_L \to e^{i\alpha_L}\psi_L$, $\psi_R \to e^{i\alpha_R}\psi_R$

1.(b)

Replace $\partial_\mu\psi_L$ by $D_\mu\psi_L$ with $\boxed{D_\mu = \partial_\mu + ieA_\mu}$

A_μ transform as $A_\mu \to A_\mu - \frac{1}{e}\partial_\mu\alpha_L(x)$

$$\boxed{\mathcal{L}_{new} = i\bar{\psi}_R\gamma^\mu\partial_\mu\psi_R + i\bar{\psi}_L\gamma^\mu D_\mu\psi_L - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}}$$

1.(c)

Covariant derivative for field φ_1 same as for electron.

$$\boxed{D_\mu^{(1)}\varphi_1 = (\partial_\mu + ieA_\mu)\varphi_1}$$

This obeys $D_\mu^{(1)}\varphi_1 \to e^{i\alpha(x)} D_\mu^{(1)}\varphi_1$ if $\varphi_1 \to e^{i\alpha(x)}\varphi_1$

Hence $|D_\mu^{(1)}\varphi_1|^2$ is gauge invariant

For field φ_2 one has $(-2)\times$ the charge of φ_1

$$\boxed{D_\mu^{(2)}\varphi_2 = (\partial_\mu - i2eA_\mu)\varphi_2}$$

$$D_\mu^{(2)}\varphi_2 \to e^{-i2\alpha(x)} D_\mu^{(2)}\varphi_2 \quad \text{if } \varphi_2 \to e^{-i2\alpha(x)}\varphi_2$$

In either case photon field always transforms as

$$A_\mu \to A_\mu - \frac{1}{e}\partial_\mu\alpha$$

Full gauge invariant Lagrangian density

$$\boxed{\mathcal{L} = \left[|D_\mu^{(1)}\varphi_1|^2 - m_1^2\varphi_1^*\varphi_1\right] + \left[|D_\mu^{(2)}\varphi_2|^2 - m_2^2\varphi_2^*\varphi_2\right] - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}}$$

2.(a)

If scalar field of charge $+e$ transforms as

$$\varphi(x) \to e^{-i\alpha(x)}\varphi(x)$$

then the other fields:

$$\begin{cases} \psi_L \to e^{i\alpha(x)}\psi_L \\ \psi_R \to e^{i2\cdot\alpha(x)}\psi_R \\ A_\mu \to A_\mu - \frac{1}{e}\partial_\mu\alpha \end{cases}$$

2. (b)

Using results of Prob. 1 one has

$$\boxed{\mathcal{L} = \left[|D_\mu \varphi|^2 - M^2 \varphi^* \varphi \right] + i \bar{\psi}_R \gamma^\mu D^R_\mu \psi_R + i \bar{\psi}_L \gamma^\mu D^L_\mu \psi_L - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}}$$

$$\boxed{D_\mu = (\partial_\mu - ieA_\mu)}\,, \quad \boxed{D^R_\mu = (\partial_\mu + i2eA_\mu)}\,, \quad \boxed{D^L_\mu = (\partial_\mu + ieA_\mu)}$$

A fermion mass term $m\bar{\psi}\psi = m(\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L)$ (see Set # 3, Prob. 3) would not be gauge invariant

2. (c)

It is easy to check that

$$\boxed{\mathcal{L}_{int} = y\,(\bar{\psi}_L \psi_R \varphi + \bar{\psi}_R \psi_L \varphi^*)}$$

(y = real coupling) is a Lorentz scalar, gauge invariant and hermitian

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3.

$$D_\mu \psi = (\partial_\mu - ig\hat{B}_\mu)\psi \to \left(\partial_\mu - igV\left[\hat{B}_\mu + \frac{i}{g}\partial_\mu\right]V^+\right)(V\psi)$$

$$= \partial_\mu (V\psi) - igV\left[\hat{B}_\mu V^+ + \frac{i}{g}(\partial_\mu V^+)\right](V\psi) =$$

$$= (\partial_\mu V)\psi + V(\partial_\mu \psi) - ig V \hat{B}_\mu \psi + V(\partial_\mu V^\dagger) V \psi$$

Use $\partial_\mu (V^\dagger V) = 0$ or $(\partial_\mu V^\dagger) V = - V^\dagger \partial_\mu V$

$$V (\partial_\mu V^\dagger) V = - \partial_\mu V$$

$$D_\mu \psi \longrightarrow \cancel{(\partial_\mu V)}\psi + V(\partial_\mu \psi) - ig V \hat{B}_\mu \psi - \cancel{(\partial_\mu V)}\psi$$

$$= V(\partial_\mu - ig \hat{B}_\mu)\psi = \boxed{V D_\mu \psi}$$

4.

Solve $\boxed{\hat{a}_H(t) = e^{i\hat{H}_s t/\hbar}\, \hat{a}_s\, e^{-i\hat{H}_s t/\hbar}}$

using $\boxed{e^{\hat{A}}\, \hat{B}\, e^{-\hat{A}} = e^{\gamma} \hat{B} \quad \text{for } [\hat{A}, \hat{B}] = \gamma \hat{B}}$

for $\hat{A} \to \frac{i\hat{H}_s t}{\hbar}$, $B \to \hat{a}_s$

$$[\hat{A}, \hat{B}] \to \left[\frac{i\hat{H}_s t}{\hbar}, \hat{a}_s\right] = \frac{it}{\hbar} \hbar\omega \underbrace{[\hat{a}_s^\dagger \hat{a}_s, \hat{a}_s]}_{= -\hat{a}_s}$$

$$= -i\omega t\, \hat{a}_s$$

$\Rightarrow \boxed{\gamma = -i\omega t}$

and $\boxed{\hat{a}_H(t) = e^{-i\omega t}\, \hat{a}_s = e^{-i\omega t}\, \hat{a}_H(t=0)}$

Obviously $\boxed{\hat{a}_H^\dagger(t) = e^{i\omega t}\, \hat{a}_s^\dagger}$

$$[\hat{a}_H(t), \hat{a}_H^\dagger(t)] = [\hat{a}_s, \hat{a}_s^\dagger] = 1$$