P8802.01 Set #5 Solutions

1.(a) Using (2) one has

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial \bar{\psi}_\alpha} = \dfrac{i}{2}\left(\gamma^\mu \partial_\mu \psi\right)_\alpha - m\psi_\alpha \\[2ex] \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi}_\alpha)} = -\dfrac{i}{2}\left(\gamma^\mu \psi\right)_\alpha \end{cases}$$

$$-\frac{i}{2}\partial_\mu(\gamma^\mu\psi)_\alpha - \left[\frac{i}{2}(\gamma^\mu\partial_\mu\psi)_\alpha - m\psi_\alpha\right] = 0$$

$$-i\gamma^\mu\partial_\mu\psi + m\psi = 0 \quad \text{or} \quad \boxed{(i\not{\partial} - m)\psi = 0}$$

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial \psi_\alpha} = -\dfrac{i}{2}\left[(\partial_\mu\bar{\psi})\gamma^\mu\right]_\alpha - m\bar{\psi}_\alpha \\[2ex] \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_\alpha)} = \dfrac{i}{2}\left(\bar{\psi}\gamma^\mu\right)_\alpha \end{cases}$$

$$\frac{i}{2}\left[(\partial_\mu\bar{\psi})\gamma^\mu\right]_\alpha - \left\{-\frac{i}{2}\left[(\partial_\mu\bar{\psi})\gamma^\mu\right]_\alpha - m\bar{\psi}_\alpha\right\} = 0$$

$$i\left[(\partial_\mu\bar{\psi})\gamma^\mu\right]_\alpha + m\bar{\psi}_\alpha = 0 \quad \text{or} \quad \boxed{i(\partial_\mu\bar{\psi})\gamma^\mu + m\bar{\psi} = 0}$$

1.(b)

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - j^\nu_{ext.}A_\nu = -\frac{1}{4}g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}F_{\mu\nu} - j^\nu_{ext.}A_\nu$$

Use $\boxed{\partial_\sigma\left(\dfrac{\partial\mathcal{L}}{\partial(\partial_\sigma A_\rho)}\right) - \dfrac{\partial\mathcal{L}}{\partial A_\rho} = 0}$ for $\rho = 0, 1, 2, 3$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\sigma A_\rho)} = -\frac{1}{4} g^{\mu\alpha} g^{\nu\beta} \frac{\partial}{\partial(\partial_\sigma A_\rho)} \left[(\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial_\mu A_\nu - \partial_\nu A_\mu) \right]$$

$$= -\frac{1}{4} g^{\mu\alpha} g^{\nu\beta} \Big\{ (\delta^\sigma_\alpha \delta^\rho_\beta - \delta^\sigma_\beta \delta^\rho_\alpha)(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

$$+ (\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\delta^\sigma_\mu \delta^\rho_\nu - \delta^\sigma_\nu \delta^\rho_\mu) \Big\}$$

$$= -\frac{1}{4} \Big\{ (g^{\mu\sigma} g^{\nu\rho} - g^{\mu\rho} g^{\nu\sigma})(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

$$+ (\partial_\alpha A_\beta - \partial_\beta A_\alpha)(g^{\sigma\alpha} g^{\rho\beta} - g^{\rho\alpha} g^{\sigma\beta}) \Big\}$$

$$= -(\partial^\sigma A^\rho - \partial^\rho A^\sigma) = -F^{\sigma\rho}$$

$$\frac{\partial \mathcal{L}}{\partial A_\rho} = -j^\rho_{ext.}$$

$$\Rightarrow \quad -\partial_\sigma F^{\sigma\rho} + j^\rho_{ext.} = 0 \qquad \text{or} \qquad \boxed{\partial_\sigma F^{\sigma\rho} = j^\rho_{ext}}$$

$\underline{\rho = 0}$

$$\begin{cases} \partial_\sigma F^{\sigma 0} = \partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} = \vec{\nabla} \cdot \vec{E} \\ j^0_{ext} = \rho_{ext} \end{cases}$$

$$\boxed{\vec{\nabla} \cdot \vec{E} = \rho_{ext}}$$

$\underline{\rho = 1}$

$$\partial_\sigma F^{\sigma 1} = \partial_0 F^{01} + \partial_2 F^{21} + \partial_3 F^{31} =$$

$$= -\frac{\partial}{\partial t} E_x + \frac{\partial}{\partial y} B_z - \frac{\partial}{\partial z} B_y$$

$$\boxed{(\vec{\nabla} \times \vec{B})_x - \frac{\partial E_x}{\partial t} = j_{x,ext.}}$$

2.(b) (Peskin & Schröder Prob. 2.1(b))

$$\mathcal{L} = -\frac{1}{4} F^{\alpha\beta} F_{\alpha\beta}$$

$$T^{\mu\nu} = \underbrace{\sum_{\sigma} \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\sigma)}}_{=-F^{\mu\sigma} \text{ from Prob. 1(b).}} \partial^\nu A_\sigma - g^{\mu\nu} \mathcal{L}$$

$$T^{\mu\nu} = -F^{\mu\sigma} \partial^\nu A_\sigma + \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}$$

$T^{\mu\nu}$ is not symmetric in $\mu \rightleftarrows \nu$. Following Peskin add a term $\partial_\sigma K^{\sigma\mu\nu}$ with $K^{\sigma\mu\nu} = F^{\mu\sigma} A^\nu$

So,

$$\hat{T}^{\mu\nu} = \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} - F^{\mu\sigma} \partial^\nu A_\sigma + \partial_\sigma (F^{\mu\sigma} A^\nu)$$

In last term use eq. of motion $\partial_\sigma F^{\mu\sigma} = 0$ (see Prob. 1)

then,

$$\hat{T}^{\mu\nu} = \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} - F^{\mu\sigma} (\partial^\nu A_\sigma - \partial_\sigma A^\nu)$$

$$\Rightarrow \boxed{\hat{T}^{\mu\nu} = \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} - F^{\mu\sigma} F^\nu{}_\sigma}$$

The last term can be written as $-\frac{1}{2}(F^{\mu\sigma} F^\nu{}_\sigma + F^\mu{}_\sigma F^{\nu\sigma})$ which is symmetric in $\mu \rightleftarrows \nu$

Next write things in terms of familiar $\vec{E}$ & $\vec{B}$ fields

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}, \quad F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$\boxed{\tilde{T}^{00} = \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} - F^{0\sigma} F^{0}{}_{\sigma}}$$

$$\begin{cases} \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} = -\frac{1}{4} F^{\alpha\beta} F_{\beta\alpha} = \frac{1}{2}\left(-\vec{E}^2 + \vec{B}^2\right) \\ -F^{0\sigma} F^{0}{}_{\sigma} = -g^{0\rho} F^{0\sigma} F_{\rho\sigma} = g^{0\rho} F^{0\sigma} F_{\sigma\rho} = F^{0\sigma} F_{\sigma 0} = \vec{E}^2 \end{cases}$$

$$\tilde{T}^{00} = \frac{1}{2}\left(-\vec{E}^2 + \vec{B}^2\right) + \vec{E}^2 = \boxed{\frac{1}{2}\left(\vec{E}^2 + \vec{B}^2\right)}$$

energy density

$$\boxed{\tilde{T}^{0k} = -F^{0\sigma} F^{k}{}_{\sigma}}$$

$$-F^{0\sigma} F^{k}{}_{\sigma} = -\sum_{j=1,2,3} \left(-E_j\right)\left(g^{k\rho} F_{\rho j}\right) = -\sum_{j} E_j F_{kj}$$

Use $F_{kj} = -\sum_i \varepsilon_{kji} B_i$

$$\Rightarrow -F^{0\sigma} F^{k}{}_{\sigma} = \sum_{j,i} E_j \,\varepsilon_{kji}\, B_i = \sum_{j,i} E_j B_i \,\varepsilon_{jik} = \left(\vec{E} \times \vec{B}\right)_k$$

$$\boxed{\tilde{T}^{0k} = \left(\vec{E} \times \vec{B}\right)_k}$$

momentum density

3(a)

Use $\gamma_5^2 = I$

$$e^{i\alpha\gamma_5} = 1 + i\alpha\gamma_5 + \frac{1}{2}\left(i\alpha\gamma_5\right)^2 + \frac{1}{3!}\left(i\alpha\gamma_5\right)^3 + \cdots$$

$$= \left(1 - \frac{\alpha^2}{2} + \cdots\right) + i\gamma_5\left(\alpha - \frac{1}{3!}\alpha^3 + \cdots\right)$$

$$= \boxed{\cos\alpha + i\sin\alpha \cdot \gamma_5}$$

Since $\{\gamma^\mu, \gamma_5\} = 0$

$$\gamma^\mu e^{i\alpha\gamma_5} = \gamma^\mu (\cos\alpha + i\sin\alpha\cdot\gamma_5) = (\cos\alpha - i\sin\alpha\cdot\gamma_5)\gamma^\mu$$

$$\boxed{\gamma^\mu e^{i\alpha\gamma_5} = e^{-i\alpha\gamma_5}\gamma^\mu}$$

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3. (b)

$$\bar\psi i\gamma^\mu(\partial_\mu\psi) \to (\psi^\dagger e^{-i\alpha\gamma_5}\gamma^0) i\gamma^\mu (\partial_\mu e^{i\alpha\gamma_5}\psi)$$

$$= \bar\psi e^{i\alpha\gamma_5} \underbrace{i\gamma^\mu e^{i\alpha\gamma_5}}_{e^{-i\alpha\gamma_5} i\gamma^\mu} \partial_\mu\psi = \bar\psi i\gamma^\mu\partial_\mu\psi$$

i.e. <u>invariant</u>

$$\begin{cases} \bar\psi\psi \to \bar\psi e^{2i\alpha\gamma_5}\psi = \cos 2\alpha\cdot\bar\psi\psi + i\sin 2\alpha\cdot\bar\psi\gamma_5\psi \\ \bar\psi\gamma_5\psi \to \bar\psi e^{2i\alpha\gamma_5}\gamma_5\psi = \cos 2\alpha\cdot\bar\psi\gamma_5\psi + i\sin 2\alpha\cdot\bar\psi\psi \end{cases}$$

Let complex field φ transform as $\varphi \to \varphi'$

then,

$$\left[(\varphi+\varphi^*)\bar\psi\psi + (\varphi-\varphi^*)\bar\psi\gamma_5\psi\right]$$

$$\Rightarrow (\varphi'+\varphi'^*)\left[\cos 2\alpha\cdot\bar\psi\psi + i\sin 2\alpha\cdot\bar\psi\gamma_5\psi\right]$$
$$+ (\varphi'-\varphi'^*)\left[i\sin 2\alpha\cdot\bar\psi\psi + \cos 2\alpha\cdot\bar\psi\gamma_5\psi\right]$$

$$= \varphi'\left[e^{i2\alpha}\bar\psi\psi + e^{i2\alpha}\bar\psi\gamma_5\psi\right]$$
$$+ \varphi'^*\left[e^{-i2\alpha}\bar\psi\psi - e^{-i2\alpha}\bar\psi\gamma_5\psi\right]$$

$\Rightarrow$ Want $\boxed{\varphi' = e^{-2i\alpha}\varphi}$ transform of φ

then $\mathcal{L}$ is invariant.

3.(c)

There will be one Noether current, since only one parameter α.

$$\delta\psi = i\gamma_5\psi \quad , \quad \delta\bar{\psi} = i\bar{\psi}\gamma_5$$

$$\delta\varphi = -2i\varphi \quad , \quad \delta\varphi^* = +2i\varphi^*$$

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}(i\gamma_5\psi) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}(-2i\varphi) + (2i\varphi^*)\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^*)}$$

$$\boxed{j^\mu = -\left(\bar{\psi}\gamma^\mu\gamma_5\psi\right) - 2i\left[(\partial^\mu\varphi^*)\varphi - \varphi^*\partial^\mu\varphi\right]}$$

3.(d)

If $\varphi = \sigma + i\pi$, then $\varphi + \varphi^* = 2\sigma$ couples to $\bar{\psi}\psi$ (scalar)

and $(\varphi - \varphi^*) = 2i\pi$ couples to $\bar{\psi}\gamma_5\psi$ (pseudo scalar)

$\Rightarrow$ Want σ field to be scalar
π field to be pseudo scalar