1.(6 pts.)
(a) The standard lagrangian density for the free Dirac system is,

$$\mathcal{L} = i\overline{\Psi}\gamma^\mu\partial_\mu\Psi \; - \; m\overline{\Psi}\Psi \qquad (1)$$

An equally valid $\mathcal{L}$ (differs by total derivative from (1)) is the more symmetric form,

$$\mathcal{L} = \frac{i}{2}\left\{\overline{\Psi}\gamma^\mu\partial_\mu\Psi \; - \; (\partial_\mu\overline{\Psi})\gamma^\mu\Psi\right\} \; - \; m\overline{\Psi}\Psi \qquad (2)$$

We saw in class that (1) leads to the usual Dirac equation for both Ψ and $\overline{\Psi}$. Verify that the same is true of (2).

(b) Derive two of Maxwell's equations from the lagrangian density,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^\nu_{ext.}A_\nu$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $j^\nu_{ext.}$ some fixed external 4-current. Write them in standard form in terms of $\vec{E}$ and $\vec{B}$ fields.

2.(6pts.)
(a) Read Peskin & Schroeder §2.2, in particular the derivation of equations [2.17], [2.18] and [2.19].

(b) Peskin & Schroeder Prob. 2.1(b).
Note that one is dealing with the pure photon system here, hence no $j^\mu_{ext.}$.

3.(8pts.)
A system is described by a complex scalar field $\phi(x)$ and a Dirac field $\Psi(x)$ with Lagrangian density.

$$\begin{aligned}\mathcal{L} &= \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi - \lambda(\phi^*\phi)^2 \\ &+ \overline{\Psi}i\gamma^\mu\partial_\mu\Psi - g[(\phi+\phi^*)\overline{\Psi}\Psi + (\phi-\phi^*)\overline{\Psi}\gamma_5\Psi]\end{aligned}$$

(a) Verify that

$$e^{i\alpha\gamma_5} \equiv \cos(\alpha) + i\sin(\alpha)\gamma_5, \qquad \gamma^\mu e^{i\alpha\gamma_5} = e^{-i\alpha\gamma_5}\gamma^\mu$$

(b) Show that the above Lagrangian is invariant under

$$\Psi \to e^{i\alpha\gamma_5}\Psi$$

provided the ϕ fields also transform appropriately. Determine what the transformation law for the ϕ fields should be. (cont'd →)

(c) What is the Noether current for this combined fermionic and bosonic transformation ?

(d) Denote the real and imaginary parts of the complex field ϕ as $\phi = \sigma + i\,\pi$. Both σ and π are real Lorentz scalar fields. If the above Lagrangian is to preserve parity, is σ a scalar or a pseudoscalar field? Is π a scalar or a pseudocalar field?