P8802.01 Set #3 Solutions

1.

Parity $\hat{P}$

$$\begin{cases} \psi(\vec{x},t) \xrightarrow{\hat{P}} \gamma_0 \psi(-\vec{x},t) \\ \bar{\psi}(\vec{x},t) \xrightarrow{\hat{P}} \psi^\dagger(-\vec{x},t)\gamma_0^\dagger \gamma_0 = \bar{\psi}(-\vec{x},t)\gamma_0 \end{cases}$$

$$\bar{\psi}\gamma^\mu\psi \xrightarrow{\hat{P}} \bar{\psi}(-\vec{x},t)\gamma_0\gamma^\mu\gamma_0\psi(-\vec{x},t)$$

$$= \begin{cases} \left(\bar{\psi}\gamma^\mu\psi\right)_{(-\vec{x},t)} & \mu = 0 \\ -\left(\bar{\psi}\gamma^\mu\psi\right)_{(-\vec{x},t)} & \mu = k \end{cases}$$

$$\bar{\psi}\gamma^\mu\gamma_5\psi \xrightarrow{\hat{P}} \bar{\psi}(-\vec{x},t)\gamma_0\gamma^\mu\gamma_5\gamma_0\psi(-\vec{x},t) =$$

$$= \begin{cases} -\left(\bar{\psi}\gamma^\mu\gamma_5\psi\right)_{(-\vec{x},t)} & \mu = 0 \\ \left(\bar{\psi}\gamma^\mu\gamma_5\psi\right)_{(-\vec{x},t)} & \mu = k \end{cases}$$

Charge Conjugation $\hat{C}$

$$\psi_{(x)} \xrightarrow{\hat{C}} i\gamma^2\psi^*_{(x)}, \qquad \bar{\psi} \xrightarrow{\hat{C}} -i\psi^{tr}\gamma^{2\dagger}\gamma^0 = i\psi^{tr}\gamma^2\gamma^0$$

Also $\boxed{\gamma^{\mu\, tr} = \begin{cases} \gamma^\mu & \mu = 0, 2 \\ -\gamma^\mu & \mu = 1, 3 \end{cases}}$, $\boxed{\gamma^{\mu\dagger} = \begin{cases} \gamma^\mu & \mu = 0 \\ -\gamma^\mu & \mu = k \end{cases}}$

$$\bar{\psi}\gamma^\mu\psi \xrightarrow{\hat{C}} -\psi^{tr}\gamma^2\gamma^0\gamma^\mu\gamma^2\psi^* = \underset{\uparrow \gamma_0\gamma_0}{\psi^\dagger}\left(\gamma^2\gamma^0\gamma^\mu\gamma^2\right)^{tr}\psi$$

$$= \bar{\psi}\gamma^0\gamma^2\gamma^{\mu\, tr}\gamma^0\gamma^2\psi = \boxed{-\bar{\psi}\gamma^\mu\psi}$$

Note: $\boxed{(\gamma^\mu)^2 = \begin{cases} I & \mu = 0 \\ -I & \mu = k \end{cases}}$

$$\bar{\Psi}\gamma^\mu\gamma_5\psi \xrightarrow{\hat{C}} \bar{\Psi}\gamma^0\gamma^2(\gamma^\mu\gamma_5)^{tr}\gamma^0\gamma^2\psi$$

$\mu = 0:$ $\quad \bar{\Psi}\gamma^0\gamma^2\gamma_5\gamma^2\psi = \bar{\Psi}\gamma^0\gamma_5\psi$

$\mu = 1:$ $\quad -\bar{\Psi}\gamma^0\gamma^2\gamma_5\gamma^1\gamma^0\gamma^2\psi = -\bar{\Psi}\gamma_5\gamma^1\psi = \bar{\Psi}\gamma^1\gamma_5\psi$

$\mu = 2:$ $\quad \bar{\Psi}\gamma^0\gamma^2\gamma_5\gamma^2\gamma^0\gamma^2\psi = \bar{\Psi}\gamma^2\gamma_5\psi$

$\mu = 3:$ $\quad -\bar{\Psi}\gamma^0\gamma^2\gamma_5\gamma^3\gamma^0\gamma^2\psi = -\bar{\Psi}\gamma_5\psi^3\psi = \bar{\Psi}\gamma^3\gamma_5\psi$

So,

$$\bar{\Psi}\gamma^\mu\gamma_5\psi \xrightarrow{\hat{C}} \boxed{\bar{\Psi}\gamma^\mu\gamma_5\psi}$$

<u>$\hat{C}\hat{P}$</u>

$$\bar{\Psi}\gamma^\mu\psi \xrightarrow{\hat{C}\hat{P}} \begin{cases} -\left(\bar{\Psi}\gamma^\mu\psi\right)_{(-\vec{x},t)} & \mu = 0 \\ +\left(\bar{\Psi}\gamma^\mu\psi\right)_{(-\vec{x},t)} & \mu = k \end{cases}$$

$$\bar{\Psi}\gamma^\mu\gamma_5\psi \xrightarrow{\hat{C}\hat{P}} \begin{cases} -\left(\bar{\Psi}\gamma^\mu\gamma_5\psi\right)_{(-\vec{x},t)} & \mu = 0 \\ +\left(\bar{\Psi}\gamma^\mu\gamma_5\psi\right)_{(-\vec{x},t)} & \mu = k \end{cases}$$

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2. (a)

$$\psi_L = P_L\psi \quad , \quad \psi_L^\dagger = \psi^\dagger P_L^\dagger = \psi^\dagger P_L$$

$$\psi_L^\dagger\gamma^0 = \psi^\dagger P_L\gamma^0 = \bar{\Psi}P_R \quad \Rightarrow \boxed{\bar{\Psi}_L = \bar{\Psi}P_R}$$

Similarly, $\quad \psi_R = P_R\psi \quad , \quad \psi_R^\dagger\gamma^0 = \psi^\dagger P_R\gamma^0 = \bar{\Psi}P_L$

$$\Rightarrow \boxed{\bar{\Psi}_R = \bar{\Psi}P_L}$$

2. (b)

In Weyl basis one has $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$, $\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, $\gamma_5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$

$$\bar{\psi} = (\psi_L^+, \psi_R^+) \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = (\psi_R^+, \psi_L^+)$$

$$\begin{cases} P_L = \frac{1}{2}\left[\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}\right] = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \\ P_R = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \end{cases}$$

$$\boxed{\psi_{\tilde{L}} = P_L \psi = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix}, \quad \psi_{\tilde{R}} = P_R \psi = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix}}$$

$$\overline{\psi_{\tilde{L}}} = \bar{\psi} P_R = (\psi_R^+, \psi_L^+) \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} = \boxed{(0, \psi_L^+)}$$

$$\overline{\psi_{\tilde{R}}} = \bar{\psi} P_L = \boxed{(\psi_R^+, 0)}$$

2. (c)

In Dirac basis $\psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_L + \psi_R \\ -\psi_L + \psi_R \end{pmatrix}$, $\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, $\gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$

$$\bar{\psi} = \frac{1}{\sqrt{2}} (\psi_L^+ + \psi_R^+, \ (-\psi_L^+ + \psi_R^+)) \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \frac{1}{\sqrt{2}} ((\psi_L^+ + \psi_R^+), \ (\psi_L^+ - \psi_R^+))$$

$$P_L = \frac{1}{2} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \quad P_R = \frac{1}{2} \begin{pmatrix} I & I \\ I & I \end{pmatrix}$$

$$\psi_{\tilde{L}} = \frac{1}{2} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_L + \psi_R \\ -\psi_L + \psi_R \end{pmatrix} = \boxed{\frac{1}{\sqrt{2}} \begin{pmatrix} \psi_L \\ -\psi_L \end{pmatrix}}$$

$$\boxed{\psi_{\tilde{R}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_R \\ \psi_R \end{pmatrix}}$$

$$\overline{\psi}_{\tilde{L}} = \frac{1}{\sqrt{2}}\left((\psi_L^\dagger + \psi_R^\dagger), \quad (\psi_L^\dagger - \psi_R^\dagger) \right) \frac{1}{2}\begin{pmatrix} I & I \\ I & I \end{pmatrix}$$

$$= \boxed{\frac{1}{\sqrt{2}}\left(\psi_L^\dagger , \psi_L^\dagger \right)}$$

$$\boxed{\overline{\psi}_{\tilde{R}} = \frac{1}{\sqrt{2}}\left(\psi_R^\dagger , -\psi_R^\dagger \right)}$$

3.

For general bilinear $\overline{\psi}\Gamma\psi$ one has

$$\overline{\psi}\Gamma\psi = \overline{\psi}\Gamma(P_L + P_R)\psi = \overline{\psi}\Gamma P_L^2\psi + \overline{\psi}\Gamma P_R^2\psi$$

S

$$\overline{\psi}\psi = (\overline{\psi}P_L)(P_L\psi) + (\overline{\psi}P_R)(P_R\psi)$$

$$= \boxed{\overline{\psi}_{\tilde{R}}\psi_{\tilde{L}} + \overline{\psi}_{\tilde{L}}\psi_{\tilde{R}}}$$

PS

$$\overline{\psi}\gamma_5\psi = (\overline{\psi}P_L)\gamma_5(P_L\psi) + (\overline{\psi}P_R)\gamma_5(P_R\psi)$$

$$= \boxed{\overline{\psi}_{\tilde{R}}\gamma_5\psi_{\tilde{L}} + \overline{\psi}_{\tilde{L}}\gamma_5\psi_{\tilde{R}}}$$

where $P_L\gamma_5 = \gamma_5 P_L$, $\quad P_R\gamma_5 = \gamma_5 P_R$

V^μ

$$\bar\psi\gamma^\mu\psi = \bar\psi\gamma^\mu P_L^2\psi + \bar\psi\gamma^\mu P_R^2\psi =$$
$$= (\bar\psi P_R)\gamma^\mu(P_L\psi) + (\bar\psi P_L)\gamma^\mu(P_R\psi) =$$
$$= \bar\psi_{\tilde L}\gamma^\mu\psi_{\tilde L} + \bar\psi_{\tilde R}\gamma^\mu\psi_{\tilde R}$$

A^μ

$$\bar\psi\gamma^\mu\gamma_5\psi = \bar\psi\gamma^\mu\gamma_5 P_L^2\psi + \bar\psi\gamma^\mu P_R^2\psi =$$
$$= (\bar\psi P_R)\gamma^\mu\gamma_5(P_L\psi) + (\bar\psi P_L)\gamma^\mu\gamma_5(P_R\psi)$$
$$= \bar\psi_{\tilde L}\gamma^\mu\gamma_5\psi_{\tilde L} + \bar\psi_{\tilde R}\gamma^\mu\gamma_5\psi_R$$

So, S & PS mix left-handed and right handed terms, while V^μ & A^μ couple only left with left and right with right.

4.(a)

$$j_1 \otimes j_2 = \tfrac{1}{2} \otimes \tfrac{1}{2} = 1 \oplus 0$$

$$j_1 \otimes j_2 \otimes j_3 = (1 \oplus 0) \otimes \tfrac{1}{2} = 1 \otimes \tfrac{1}{2} + \tfrac{1}{2} =$$
$$= \tfrac{3}{2} \oplus \left(\tfrac{1}{2}\right)_S \oplus \left(\tfrac{1}{2}\right)_A$$

$$|0\rangle_{j_1 j_2} = \frac{1}{\sqrt{2}}\left[|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2\right]$$

$$\Rightarrow \chi(M_A) \equiv \left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle^{(A)}_{j_1 j_2 j_3} = \boxed{\frac{1}{\sqrt{2}}\left[|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2\right]|\uparrow\rangle_3}$$

$$\begin{cases} |1,1\rangle_{j_1 j_2} = |\uparrow\rangle_1 |\uparrow\rangle_2 , \quad |1,0\rangle_{j_1 j_2} = \frac{1}{\sqrt{2}}\left[|\uparrow\rangle_1 |\downarrow\rangle_2 + |\downarrow\rangle_1 |\uparrow\rangle_2 \right] \\ |1,-1\rangle_{j_1 j_2} = |\downarrow\rangle_1 |\downarrow\rangle_2 \end{cases}$$

$$\chi(M_S) \equiv |\tfrac{1}{2},\tfrac{1}{2}\rangle^{(S)}_{j_1 j_2 j_3} = a\, |1\,1\rangle_{j_1 j_2} |\downarrow\rangle_3 + b\, |1\,0\rangle_{j_1 j_2} |\uparrow\rangle_3$$

Looking up C-G. coefficients $a = \sqrt{\frac{2}{3}}$, $b = -\frac{1}{\sqrt{3}}$

$$\chi(M_S) = \sqrt{\frac{2}{3}} \left[|\uparrow\rangle_1 |\uparrow\rangle_2 \right] |\downarrow\rangle_3 - \frac{1}{\sqrt{3}} \frac{1}{\sqrt{2}} \left[|\uparrow\rangle_1 |\downarrow\rangle_2 + |\downarrow\rangle_1 |\uparrow\rangle_2 \right] |\uparrow\rangle_3$$

$$= \boxed{\frac{1}{\sqrt{6}} \left[2\, |\uparrow\rangle_1 |\uparrow\rangle_2 |\downarrow\rangle_3 - |\uparrow\rangle_1 |\downarrow\rangle_2 |\uparrow\rangle_3 - |\downarrow\rangle_1 |\uparrow\rangle_2 |\uparrow\rangle_3 \right]}$$

4.(b) repeating the calculations of 4.(a) in isospin space means $|\uparrow\rangle \to |u\rangle$, $|\downarrow\rangle \to |d\rangle$

So,

$$\boxed{|P\rangle_A = \frac{1}{\sqrt{2}} \left[|udu\rangle - |duu\rangle \right]}$$

$$\boxed{|P\rangle_S = \frac{1}{\sqrt{6}} \left[2\,|uud\rangle - |udu\rangle - |duu\rangle \right]}$$

For a proton state we want it to be totally symmetric in spin ⊗ isospin ⊗ real space
(Note: color part will be totally antisymmetric)

$$|P,\uparrow\rangle = \frac{1}{\sqrt{2}} \left[|P\rangle_A\, \chi(M_A) + |P\rangle_S\, \chi(M_S) \right]$$

$$= \frac{1}{\sqrt{2}} \Big\{ \frac{1}{2} \left(|udu\rangle - |duu\rangle \right) \left(|\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle \right) + \frac{1}{6} \left[2\,|uud\rangle - |udu\rangle - |duu\rangle \right] \left[2\,|\uparrow\uparrow\downarrow\rangle - |\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle \right] \Big\} =$$

$$= \frac{1}{\sqrt{2}} \Big\{ \frac{1}{3} |uud\rangle \left[2|\uparrow\uparrow\downarrow\rangle - |\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle \right]$$

$$+ |udu\rangle \left[\frac{1}{2}\left(|\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle\right) - \frac{1}{6}\left(2|\uparrow\uparrow\downarrow\rangle - |\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle\right) \right]$$

$$+ |duu\rangle \left[\frac{1}{2}\left(|\downarrow\uparrow\uparrow\rangle - |\uparrow\downarrow\uparrow\rangle\right) - \frac{1}{6}\left(2|\uparrow\uparrow\downarrow\rangle - |\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle\right) \right] \Big\}$$

$$= \boxed{\begin{aligned} \frac{1}{\sqrt{18}} \Big\{ & |uud\rangle \left[2|\uparrow\uparrow\downarrow\rangle - |\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle \right] \\ & + |udu\rangle \left[2|\uparrow\downarrow\uparrow\rangle - |\uparrow\uparrow\downarrow\rangle - |\downarrow\uparrow\uparrow\rangle \right] \\ & + |duu\rangle \left[2|\downarrow\uparrow\uparrow\rangle - |\uparrow\uparrow\downarrow\rangle - |\uparrow\downarrow\uparrow\rangle \right] \Big\} \end{aligned}}$$

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5. In iso-spin space the iso-singlet boundstate of an iso-doublet of quarks $\binom{u}{d}$ and an iso-doublet of antiquarks $(\bar{u}, \bar{d})_{\bar{2}} \equiv (-\bar{d}, \bar{u})_2$, is given by

$$\frac{1}{\sqrt{2}}\left(|u\bar{u}\rangle + |d\bar{d}\rangle\right) \qquad \text{(iso-space)}$$

In spin space for $S=1$ one has

$$|\uparrow\uparrow\rangle, \quad \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right), \quad |\downarrow\downarrow\rangle \qquad \text{(spin-space)}$$

and in color space the color-singlet combination is

$$\frac{1}{\sqrt{3}}\left(R\bar{R} + G\bar{G} + B\bar{B}\right) \Rightarrow \frac{1}{\sqrt{3}} \sum_{c=1}^{3} (c\bar{c})$$

$$\Rightarrow \boxed{\begin{aligned} |\omega, S_z=1\rangle &= \frac{1}{\sqrt{6}} \sum_c \left(|u^c \bar{u}^c\rangle + |d^c \bar{d}^c\rangle\right) |\uparrow\uparrow\rangle \\ |\omega, S_z=0\rangle &= \frac{1}{\sqrt{12}} \sum_c \left(|u^c \bar{u}^c\rangle + |d^c \bar{d}^c\rangle\right)\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \\ |\omega, S_z=-1\rangle &= \frac{1}{\sqrt{6}} \sum_c \left(|u^c \bar{u}^c\rangle + |d^c \bar{d}^c\rangle\right) |\downarrow\downarrow\rangle \end{aligned}}$$