Physics 8802.01 Set #2 Solutions

1. (a)

Obviously $[S^{\mu\nu}, P^\rho] = 0$

So, if $F = F(x)$ is an arbitrary fcn. of x^μ, one has

$$[M^{\mu\nu}, P^\rho]F = \Big(i(x^\mu\partial^\nu - x^\nu\partial^\mu)(i\partial^\rho)\Big)F - \Big((i\partial^\rho)\, i(x^\mu\partial^\nu - x^\nu\partial^\mu)\Big)F \quad (*)$$

$$2^{nd} \text{ term in } (*) = \frac{\partial}{\partial x_\rho}\Big([g^{\mu\sigma}x_\sigma\partial^\nu - g^{\nu\sigma}x_\sigma\partial^\mu]F\Big)$$

$$= (g^{\mu\sigma}\delta^\rho_\sigma\,\partial^\nu - g^{\nu\sigma}\delta^\rho_\sigma\,\partial^\mu)F + \underbrace{\big((x^\mu\partial^\nu - x^\nu\partial^\mu)\partial^\rho\big)F}_{\text{cancels } 1^{st} \text{ term in } (*)}$$

$$\Rightarrow \boxed{[M^{\mu\nu}, P^\rho] = -i(g^{\mu\rho}P^\nu - g^{\nu\rho}P^\mu)}$$

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1. (b)

$$[M^{\mu\nu}, P^\rho P_\rho] = [M^{\mu\nu}, P^\rho]P_\rho + P^\rho\underbrace{[M^{\mu\nu}, P_\rho]}_{= P^\sigma g_{\sigma\rho}}$$

$$= -i(g^{\mu\rho}P^\nu - g^{\nu\rho}P^\mu)P_\rho + P^\rho g_{\sigma\rho}(-i)(g^{\mu\sigma}P^\nu - g^{\nu\sigma}P^\mu)$$

$$= -i(P^\mu P^\nu - P^\nu P^\mu + P^\mu P^\nu - P^\nu P^\mu) = \boxed{\emptyset}$$

2. Apply $\hat{\Lambda}_D^{(Dirac)}$ from Set #1 to the rest frame solutions

$$\hat{\Lambda}_D^{(Dirac)} = \sqrt{\frac{E_p+m}{2m}} \begin{pmatrix} I & \frac{\vec{p}\cdot\vec{\sigma}}{E_p+m} \\ \frac{\vec{p}\cdot\vec{\sigma}}{E_p+m} & I \end{pmatrix}$$

Then,

$$\begin{cases} u(\vec{p},s) = \sqrt{E_p+m} \begin{pmatrix} \chi_s \\ \frac{\vec{p}\cdot\vec{\sigma}}{E_p+m}\chi_s \end{pmatrix} \\ v(\vec{p},s) = \sqrt{E_p+m} \begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}}{E_p+m}\xi_s \\ \xi_s \end{pmatrix} \end{cases} \qquad (A)$$

Let $P_\pm = P_1 \pm iP_2$, $P_z = P_3$ $\Rightarrow$ $\vec{p}\cdot\vec{\sigma} = \begin{pmatrix} P_z & P_- \\ P_+ & -P_z \end{pmatrix}$

$$\Rightarrow \begin{cases} \psi^{(1)}_{(x)} = \sqrt{E_p+m} \begin{pmatrix} 1 \\ 0 \\ P_z/(E_p+m) \\ P_+/(E_p+m) \end{pmatrix} e^{-ip\cdot x}, \quad \psi^{(2)}_{(x)} = \sqrt{E_p+m} \begin{pmatrix} 0 \\ 1 \\ P_-/(E_p+m) \\ -P_z/(E_p+m) \end{pmatrix} e^{-ip\cdot x} \\ \psi^{(3)}_{(x)} = \sqrt{E_p+m} \begin{pmatrix} P_-/(E_p+m) \\ -P_z/(E_p+m) \\ 0 \\ 1 \end{pmatrix} e^{ip\cdot x}, \quad \psi^{(4)}_{(x)} = \sqrt{E_p+m} \begin{pmatrix} -P_z/(E_p+m) \\ -P_+/(E_p+m) \\ -1 \\ 0 \end{pmatrix} e^{ip\cdot x} \end{cases} \qquad (B)$$

3(a)

$$\left[\sum_\mu \gamma^\mu (i\partial_\mu - eA_\mu) - m\right]\psi = 0 \qquad (1)$$

Complex conjugate (1) and multiply from left by γ^2

$$\left[\sum_\mu \gamma^2(\gamma^\mu)^* (-i\partial_\mu - eA_\mu) - \gamma^2 m\right]\psi^* = 0$$

$\uparrow$ insert $I = -(\gamma^2)^2$

$$\left[\sum_\mu \gamma^2 (\gamma^\mu)^* \gamma^2 (i\partial_\mu + eA_\mu) + \underbrace{\gamma^2 m \gamma^2}_{= -m}\right](\gamma^2\psi^*) = 0$$

$(\gamma^\mu)^* = \gamma^\mu$ for $\mu = 0, 1, 3$, $(\gamma^2)^* = -\gamma^2$

$\Rightarrow \gamma^2 (\gamma^\mu)^* \gamma^2 = -\gamma^2 \cdot \gamma^2 \gamma^\mu = \gamma^\mu$

So, finally $\left[\sum_\mu \gamma^\mu (i\partial_\mu + eA_\mu) - m\right](\gamma^2\psi^*) = 0$

$\boxed{\psi_c = i\gamma^2\psi^* \text{ obeys } (2)}$, i.e. an equation for a particle with same mass as the electron but with opposite charge. Hence ψ_c describes a positron.

3(b)

Start from $\psi^{(s=1,2)} = u(p,s)\, e^{-ip\cdot x}$ and use

$$\gamma^2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}$$

$$\psi_c^{(s=1,2)} = i\gamma^2 \left(u(p,s)\, e^{-ip\cdot x}\right)^* = i\, e^{ip\cdot x} \sqrt{E_p+m} \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \begin{pmatrix} \chi_s \\ \frac{\vec{p}\cdot\vec{\sigma}}{E_p+m}\chi_s \end{pmatrix}^*$$

$$= e^{ip\cdot x}\sqrt{E_p+m}\begin{pmatrix} i\sigma_2 \dfrac{(\vec{p}\cdot\vec{\sigma})^*}{E_p+m}\chi_s^* \\ -i\sigma_2\chi_s^* \end{pmatrix}$$

Recall $\chi_s^* = \chi_s$, $-i\sigma_2\chi_s = \xi_s$

Also $i\sigma_2(\sigma_i)^* = \sigma_i(-i\sigma_2)$ or $i\sigma_2(\vec{p}\cdot\vec{\sigma})^* = (\vec{p}\cdot\vec{\sigma})(-i\sigma_2)$

$$\Rightarrow \boxed{\psi_c^{(s=1,2)}(x) = \sqrt{E_p+m}\begin{pmatrix} \dfrac{\vec{p}\cdot\vec{\sigma}}{E_p+m}\xi_s \\ \xi_s \end{pmatrix} e^{ip\cdot x} = \psi^{(s+2)}(x)}$$

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4. (H. & M. Exc. 5.9)

$$u(p,s) = N\begin{pmatrix}\chi_s \\ \dfrac{\vec{p}\cdot\vec{\sigma}}{D}\chi_s\end{pmatrix}, \quad \bar{u}(p,s) = u^\dagger\gamma^0 = N^*\left(\chi_s^\dagger,\ \chi_s^\dagger\frac{-\vec{p}\cdot\vec{\sigma}}{D}\right)$$

with $N = \sqrt{E_p+m}$, $D = E_p+m$

$$\sum_s u(p,s)\,\bar{u}(p,s) = N^2\sum_s\begin{pmatrix}\chi_s\chi_s^\dagger & \chi_s\chi_s^\dagger\dfrac{-\vec{p}\cdot\vec{\sigma}}{D} \\ \dfrac{\vec{p}\cdot\vec{\sigma}}{D}\chi_s\chi_s^\dagger & \dfrac{\vec{p}\cdot\vec{\sigma}}{D}\chi_s\chi_s^\dagger\dfrac{-\vec{p}\cdot\vec{\sigma}}{D}\end{pmatrix}$$

$$\sum_s\chi_s\chi_s^\dagger = \begin{pmatrix}1\\0\end{pmatrix}(1,0) + \begin{pmatrix}0\\1\end{pmatrix}(0,1) = \begin{pmatrix}1&0\\0&0\end{pmatrix} + \begin{pmatrix}0&0\\0&1\end{pmatrix} = I$$

$$\sum_s u(p,s)\,\bar{u}(p,s) = (E_p+m)\begin{pmatrix} I & -\vec{p}\cdot\vec{\sigma}/D \\ \dfrac{\vec{p}\cdot\vec{\sigma}}{D} & -\dfrac{(\vec{p}\cdot\vec{\sigma})^2}{D^2}\end{pmatrix}$$

$$-\frac{(\vec{p}\cdot\vec{\sigma})^2}{D^2} = -\frac{\vec{p}^{\,2}}{(E_p+m)^2} = -\frac{E_p^2-m^2}{(E_p+m)^2} = -\frac{E_p-m}{E_p+m}$$

$$\sum_s u(p,s)\,\bar{u}(p,s) = \begin{pmatrix} E_p+m & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -E_p+m \end{pmatrix} = E_p\gamma^0 - \vec{p}\cdot\vec{\gamma} + m = \boxed{\not{p}+m}$$

Similarly,

$$v(p,s) = N\begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}}{D}\xi_s \\ \xi_s \end{pmatrix}, \quad \bar{v}(p,s) = N^*\left(\xi_s^\dagger \frac{\vec{p}\cdot\vec{\sigma}}{D}, -\xi_s^\dagger\right)$$

$$\sum_s v(p,s)\,\bar{v}(p,s) = |N|^2 \sum_s \begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}}{D}\xi_s\xi_s^\dagger\frac{\vec{p}\cdot\vec{\sigma}}{D} & -\frac{\vec{p}\cdot\vec{\sigma}}{D}\xi_s\xi_s^\dagger \\ \xi_s\xi_s^\dagger\frac{\vec{p}\cdot\vec{\sigma}}{D} & -\xi_s\xi_s^\dagger \end{pmatrix}$$

$$\sum_s \xi_s\xi_s^\dagger = \begin{pmatrix} 0 \\ 1 \end{pmatrix}(0, 1) + \begin{pmatrix} -1 \\ 0 \end{pmatrix}(-1, 0) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = I$$

$$\sum_s v(p,s)\,\bar{v}(p,s) = (E_p+m)\begin{pmatrix} \frac{\vec{p}^{\,2}}{D^2} & -\frac{\vec{p}\cdot\vec{\sigma}}{D} \\ \frac{\vec{p}\cdot\vec{\sigma}}{D} & -I \end{pmatrix} = \begin{pmatrix} E_p-m & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -E_p-m \end{pmatrix}$$

$$= \boxed{\not{p}-m}$$

5. $$\hat{C}\psi_{Maj.} = i\gamma^2\psi^*_{Maj} = \begin{pmatrix} 0 & i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} \psi_L^* \\ i\sigma^{2*}\psi_L \end{pmatrix}$$

$$= \begin{pmatrix} \psi_L \\ -i\sigma^2\psi_L^* \end{pmatrix} = \psi_{Maj}$$

So, a Majorana spinor is equal to its charge conjugate. In other words, a particle described by a Majorana spinor must be equal to its anti-particle.
As such it must be a neutral particle.