Physics 8802.01 Set #1 Solutions

1.(b)

Use $c = 2.9979 \times 10^{8}\ m/s$, $\hbar = 1.0545 \times 10^{-34}\ J\cdot sec$

$$\hbar c = \frac{(2.9979 \times 10^{8})(1.0545 \times 10^{-34})\ J\cdot m}{1.602 \times 10^{-19}\ J/eV} =$$

$$= 1.973 \times 10^{-7}\ eV\cdot m = 1.973 \times 10^{8}\ eV\cdot fm = 0.1973\ GeV\cdot fm$$

For natural units with $\hbar = 1$, $c = 1$ one then has

$$1 = \hbar c = 0.1973\ GeV\cdot fm$$

$$\Rightarrow \boxed{1\ fm = \frac{1}{0.1973\ GeV} = 5.068\ GeV^{-1}} \quad (1)$$

$$1\ mb = 10^{-3}\ b = 10^{-27}\ cm^{2}$$

From (1) $\Rightarrow$ $(1\ fm)^2 = (5.068\ GeV^{-1})^2 = 25.68\ GeV^{-2}$

$$\boxed{(1\ GeV)^{-2}} = 0.0389\ fm^{2} = 0.0389 \times 10^{-26}\ cm^{2} = \boxed{0.389\ mb}$$

2.(a)(i)

A boost in the z-direction is given by

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \cosh\eta & 0 & 0 & -\sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix}^{\mu}{}_{\nu}$$

For infinitesimal η, $\cosh\eta \approx 1$, $\sinh\eta \approx \eta$

$$\Lambda^{\mu}{}_{\nu} \approx \delta^{\mu}{}_{\nu} + \underbrace{\begin{pmatrix} 0 & 0 & 0 & -\eta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\eta & 0 & 0 & 0 \end{pmatrix}^{\mu}{}_{\nu}}_{\equiv \, \varepsilon^{\mu}{}_{\nu}}$$

$$\varepsilon^{\mu\nu} = \varepsilon^{\mu}{}_{\sigma} g^{\sigma\nu} = \begin{pmatrix} 0 & 0 & 0 & \eta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\eta & 0 & 0 & 0 \end{pmatrix}^{\mu\nu} \qquad (2)$$

$$\varepsilon_{\mu\nu} = g_{\mu\sigma} \varepsilon^{\sigma}{}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & -\eta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \eta & 0 & 0 & 0 \end{pmatrix}_{\mu\nu} \qquad (3)$$

Obviously $\varepsilon^{\nu\mu} = -\varepsilon^{\mu\nu}$, $\varepsilon_{\nu\mu} = -\varepsilon_{\mu\nu}$

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2. (a) (ii)

A rotation about the x-axis is given by

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix}^{\mu}{}_{\nu} \approx \delta^{\mu}{}_{\nu} + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta \\ 0 & 0 & -\theta & 0 \end{pmatrix}^{\mu}{}_{\nu}}_{\varepsilon^{\mu}{}_{\nu}}$$

As with (2) & (3) one finds

$$\varepsilon^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\theta \\ 0 & 0 & \theta & 0 \end{pmatrix}^{\mu\nu}$$

$$\varepsilon_{\mu\nu} = \begin{pmatrix} & \text{same} & \end{pmatrix}_{\mu\nu}$$

Again $\varepsilon^{\nu\mu} = -\varepsilon^{\mu\nu}$, $\varepsilon_{\nu\mu} = -\varepsilon_{\mu\nu}$

2.(b)

For an infinitesimal L.T. a scalar field transforms as

$$\varphi' = \varphi(\hat{\Lambda}^{-1}x) = \varphi(x+\delta x) \approx \varphi(x) + \delta x^{\mu}\,\partial_\mu \varphi$$

For $\hat{\Lambda}^{-1}$ one has $\delta x^{\mu} = -\varepsilon^{\mu}{}_{\nu}\, x^{\nu}$

Hence,

$$\varphi' = \varphi(x) - \varepsilon^{\mu}{}_{\nu}\, x^{\nu}\partial_\mu \varphi = \varphi(x) - \varepsilon^{\mu\nu} x_\nu \partial_\mu \varphi =$$

$$= \varphi(x) - \frac{i}{2}\varepsilon^{\mu\nu}(-i)\left(x_\nu \partial_\mu - x_\mu \partial_\nu\right)$$

use $\varepsilon^{\nu\mu} = -\varepsilon^{\mu\nu}$

Comparing with $\varphi' = \left(\hat{I} - \frac{i}{2}\varepsilon^{\mu\nu}\hat{M}_{\mu\nu}\right)\varphi$ one finds

$$\boxed{M_{\mu\nu} = i\left(x_\mu \partial_\nu - x_\nu \partial_\mu\right)}$$

3. Under infinitesimal rotations $\vec{\theta}$ & boosts $\vec{\eta}$ one has

$$\begin{cases} \psi_L \to \left[1 + i\frac{\vec{\sigma}\cdot\vec{\theta}}{2} + \frac{\vec{\sigma}}{2}\cdot\vec{\eta}\right]\psi_L & (4) \\ \psi_R \to \left[1 + i\frac{\vec{\sigma}\cdot\vec{\theta}}{2} - \frac{\vec{\sigma}}{2}\cdot\vec{\eta}\right]\psi_R & (5) \end{cases}$$

Then

$$-i\sigma^2\psi_L^* \to -i\sigma^2\left[1 - i\frac{\vec{\sigma}^*\cdot\vec{\theta}}{2} + \frac{\vec{\sigma}^*}{2}\cdot\vec{\eta}\right]\psi_L^*$$

$(\sigma^2)^2 = I$

Use $\sigma^2\sigma^{i*}\sigma^2 = -\sigma^i$

then

$$-i\sigma^2\psi_L^* \to \left[1 + i\frac{\vec{\sigma}\cdot\vec{\theta}}{2} - \frac{\vec{\sigma}\cdot\vec{\eta}}{2}\right]\left(-i\sigma^2\psi_L^*\right)$$

and one sees that $\left(-i\sigma^2\psi_L^*\right)$ transform like ψ_R in (5).

4. (a)

Frame S' must be moving in $-\vec{p}$ direction relative to frame S with rapidity η obeying

$$E_p = m\gamma = m\cosh\eta \qquad , \qquad p \equiv |\vec{p}| = m|\vec{v}|\gamma = m\sinh\eta$$

$$\Rightarrow e^{\eta} = \frac{E_p + p}{m} \quad \Rightarrow \quad \boxed{\begin{array}{c} \vec{\eta} = \ln\left(\frac{E_p+p}{m}\right)\cdot\hat{n} \\ \hline \hat{n} = -\hat{p} = -\frac{\vec{p}}{|\vec{p}|} \end{array}}$$

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4. (b)

Using identity $\sigma^i\sigma^j = i\,\varepsilon_{ijk}\,\sigma^k + \delta_{ij}\,I$

one finds

$$(\vec{\eta}\cdot\vec{\sigma})^2 = \eta^2 I \qquad I = 2\times 2 \text{ identity matrix}$$

Taylor expanding $e^{\pm\frac{1}{2}\vec{\eta}\cdot\vec{\sigma}}$ gives

$$e^{\frac{1}{2}\vec{\eta}\cdot\vec{\sigma}} = I\cosh\frac{\eta}{2} + \hat{n}\cdot\vec{\sigma}\,\sinh\frac{\eta}{2}$$

$$e^{-\frac{1}{2}\vec{\eta}\cdot\vec{\sigma}} = I\cosh\frac{\eta}{2} - \hat{n}\cdot\vec{\sigma}\,\sinh\frac{\eta}{2}$$

$$\Rightarrow \boxed{\hat{\Lambda}_D^{(\text{Weyl})} = \begin{pmatrix} I\cosh\frac{\eta}{2} + \hat{n}\cdot\vec{\sigma}\,\sinh\frac{\eta}{2} & 0 \\ 0 & I\cosh\frac{\eta}{2} - \hat{n}\cdot\vec{\sigma}\,\sinh\frac{\eta}{2} \end{pmatrix}}$$

4. (c) $\hat{\Lambda}_D^{(Dirac)} = U \hat{\Lambda}_D^{(Weyl)} U^\dagger$

Let $A \equiv I \cosh\frac{\eta}{2}$, $B \equiv \sinh\frac{\eta}{2}$

then $\hat{\Lambda}_D^{(Dirac)} = \begin{pmatrix} A & -\hat{n}\cdot\vec{\sigma}\, B \\ -\hat{n}\cdot\vec{\sigma}\, B & A \end{pmatrix}$

$$\boxed{\hat{\Lambda}_D^{(Dirac)} = \begin{pmatrix} I \cosh\frac{\eta}{2} & -\hat{n}\cdot\vec{\sigma}\sinh\frac{\eta}{2} \\ -\hat{n}\cdot\vec{\sigma}\sinh\frac{\eta}{2} & I\cosh\frac{\eta}{2} \end{pmatrix}}$$

Note: $-\hat{n} = \hat{p}$

4. (d)

From part (a) $\cosh\eta = \frac{E_p}{m}$, $\sinh\eta = \frac{p}{m}$

Also $\cosh^2\frac{\eta}{2} = \frac{\cosh\eta + 1}{2}$, $\sinh\frac{\eta}{2}\cdot\cosh\frac{\eta}{2} = \frac{1}{2}\sinh\eta$

$\Rightarrow \cosh\frac{\eta}{2} = \sqrt{\frac{E_p+m}{2m}}$, $\sinh\frac{\eta}{2} = \frac{p}{2m}\frac{\sqrt{2m}}{\sqrt{E_p+m}} = \sqrt{\frac{E_p+m}{2m}}\cdot\frac{p}{E_p+m}$

$$\boxed{\hat{\Lambda}_D^{(Dirac)} = \sqrt{\frac{E_p+m}{2m}}\begin{pmatrix} I & \frac{\vec{p}\cdot\vec{\sigma}}{E_p+m} \\ \frac{\vec{p}\cdot\vec{\sigma}}{E_p+m} & I \end{pmatrix}}$$