Plan: Recap and loose ends from previous notes.

As #10 Problem 6 is a good problem for checking your physical understanding of Green functions and also what happens with conductors.

We are asked to use the physical interpretation of the Dirichlet Green function $G_0(x,x')$ to show that $\Delta G(x,x') < 0$, where

$$G_0(x,x') = \frac{1}{4\pi \varepsilon_0 |x-x'| + \lambda(x,x')} \frac{1}{|x-x'|} e^{-\lambda(x,x')}$$

The picture at the right describes a possible $V$, where $x'$ allows for strange shapes and disconnected surfaces (like $S'$).

The physical interpretation is that $G_0(x,x')$ is the scalar potential at $x'$ due to a point charge at $x$ in the presence of a perfect, grounded conductor at the surface $S$ of the volume $V$ (including $S'$ as part of the surface).

This conforms to $G_0$ being zero on the surface ("grounded")

So now we ask: what is the induced charge density $\sigma(x)$ on the surface(s)? Claim: it is negative (or zero) everywhere.

- Why couldn’t it be positive in some remote part? Think of field lines for the electric field. They start on positive charges and end on negative charges. If there is a region of positive $\sigma$, then the field line in $V$ from here must also end on the surface. If so, then $\mathbf{E} = 0$ along this line is positive definite, but also equal to the potential difference, which is zero (perfect grounded conductor) \(\Rightarrow\) contradiction

\(\Rightarrow\) the surface charge is negative or zero.

But if the change is negative, the scalar potential is also negative everywhere (e.g., from the integral for the potential).

For part b) see the solutions to Problem 10.
Another follow-up to Problem 10, this time to problem 3.

Here the issue is calculating

\[ \int_0^1 p_i(x) \, dx \]

see (33) for options. One is recursion relations, where do they come from?

As a set \( \mathcal{G} \), we return to (33) and its expansion for the free Green function:

\[ G(x, y) = \sum_{n=0}^{\infty} \frac{\lambda_n e^{-\lambda_n r}}{\lambda_n^3} p_n(\cos \theta) \]

which can apply in unexpected ways. An example is on (33), which is to find the potential anywhere due to a disk of charge \( q \) and radius \( a \) at height \( b \) above the origin.

We know that we have the expansion (in spherical coords):

\[ \Phi(r, \theta) = \sum_{n=1}^{\infty} (A_n r^n + B_n r^{-n}) p_n(\cos \theta) \]

but how do we find the coefficients?

The first key is to use the symmetry to solve a special case problem: here \( r \) on the \( z \)-axis \( \Rightarrow \theta = 0 \)

\[ \Rightarrow \Phi(r, 0) = \frac{q}{4\pi \varepsilon_0} \frac{1}{r^2 - r^2_0 + 2 rc \cos \alpha} \] (eqn from the basic integration)

The second key is to recognize this is the same as a free Green function

\[ G_{\text{free}}(x, y) = \frac{\lambda_n e^{-\lambda_n r}}{\lambda_n^3} p_n(\cos \theta) \]

with \( c \) taking the place of \( r' \) and \( y \to \alpha_c \).

\[ \Rightarrow \Phi(r, 0) = \frac{q}{4\pi \varepsilon_0} \sum_{n=0}^{\infty} \frac{\lambda_n e^{-\lambda_n r}}{\lambda_n^3} p_n(\cos \theta) \]

where \( r_0 = \min \{ r, C \} \) and \( \alpha_c = \max \{ \alpha, r_c \} \)

So finally

\[ \Phi(r, \theta) = \frac{q}{4\pi \varepsilon_0} \sum_{n=0}^{\infty} \frac{\lambda_n e^{-\lambda_n r}}{\lambda_n^3} p_n(\cos \theta) p_1(\cos \theta) \]

which works everywhere (whether \( r < C \) or \( r > C \)),
Now for recursion relations, we use a generating function for Legendre polynomials, as detailed on 229-230.

The key to establishing the generating function is to recognize that another basic problem: A point potential due to a point charge away from the origin (figure at left) can use the expansion \( \frac{\sin x}{x} \) as well.

Note the results we can get:

- \( P_l(1) = 1 \), \( P_l(-1) = (-1)^l \) for all \( l \) at the same time!
- Recursion relations with and without derivatives:
  - \( P_l(-x) = (-1)^l P_l(x) \)
  - \( P_l(\cos \theta) \leq P_l(1) = 1 \)

Next let's extend the expansion in Legendre polynomials, which applies for \( \Theta(x) \) when we have azimuthal symmetry. (so the expansion is in \( P_l(\cos \theta) \) or \( 2 \begin{pmatrix} x & x' \end{pmatrix} \), where the expansion is in \( P_l(\cos \varphi) \), where \( \varphi \) is the angle between \( \mathbf{x} \) and \( \mathbf{x}' \).

- We would like to use \( 0, \dot{\varphi} \) of \( \varphi \), \( x \) of \( x \), \( y \) of \( y \), \( z \) of \( z \).

  \( \Rightarrow \) use the addition theorem:

  \[ P_l(\cos \varphi) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y^m_l(\hat{\varphi}) Y^m_l(\hat{\vartheta}) \]

  Note that \( P_l(\cos \theta) \) real \( \Rightarrow \) we can put \( P_l \) on either \( Y^m_l \),

  \[ \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \sum_{l'} \sum_{m'} \frac{\delta_{ll'}}{\delta_{mm'}} \sum_{l''} \sum_{m''} \frac{\delta_{ll''}}{\delta_{mm''}} Y^m_l(\hat{\varphi}) Y^m_l(\hat{\vartheta}) \]

  [see 237 for other expansions]

The \( Y^m_l \)'s (or \( Y^m_l \)'s), called "spherical harmonics" arise when we do the full separation of variables in spherical coordinates,

\[ \nabla^2 \Phi(r, \theta, \phi) = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Phi}{\partial \phi^2} \]

where \( \nabla^2 = \frac{\hbar^2}{m^2} \) from quantum mechanics,

which implies

\[ \frac{1}{\hbar^2} \nabla^2 \Phi = -i \frac{\partial}{\partial \phi} \Rightarrow \frac{1}{\hbar^2} Z \Phi = m \Phi \quad \text{and} \quad \frac{1}{\hbar^2} Z^2 \Phi = \ell (\ell + 1) \Phi \]

So the \( Y_{lm} \) are eigenstates of both \( L^2 \) and \( L_z \).

They form a complete set for expanding any function of \( \theta \) and \( \phi \).

For \( l=0, m=0 \), \( Y_{00} = \frac{1}{\sqrt{4\pi}} \) is independent of angle.

It is normalized. In general,

\[
\int_0^{2\pi} \int_0^\pi Y_{lm}^* (\theta, \phi) Y_{lm} (\theta, \phi) = \delta_{ll'} \delta_{mm'}
\]

For \( l=1 \) \( Y_m \)'s times \( r \) given \( \theta, \phi \), and \( y \):

\[
Y_{10} = \sqrt{\frac{3}{4\pi}} \cos \theta = \sqrt{\frac{3}{4\pi}} r \\
Y_{11} = -\sqrt{\frac{3}{2\pi}} \sin \theta e^{i\phi} \\
Y_{1-1} = -\sqrt{\frac{3}{2\pi}} \sin \theta e^{-i\phi}
\]

\[
\Rightarrow r(Y_{11} + Y_{1-1}) \propto [x]
\]

An application of the expansion with spherical harmonics for \( \phi(\theta, \phi) \) is given on (23.9) - (23.16). The general expansion is

\[
\Phi(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (A_{lm} r^l + B_{lm} r^{-l-1}) Y_{lm}(\theta, \phi)
\]

where \( A_{lm} \) and \( B_{lm} \) are determined by BC's, e.g., regular at origin or going to zero at \( \infty \) along with projection.

\[
\int_0^\pi \int_0^{2\pi} r^l Y_{lm}^* (\theta, \phi) \Phi(\theta, \phi) = A_{lm} n^l + B_{lm} n^{-l-1}
\]