''put hypergeometric on board'': \( x^y - xy + 2xy - \frac{1}{2}y^2/y + y = 0 \)

- take \( y = x^p \) \( a_n = 0 \), \( x^r \) \( p \in \mathbb{R} \)

\[ p = \frac{1}{2}, y = x^{2n} \]

\[ y_n = \sum_{n=0}^{\infty} a_n z^n \]

\[ \Rightarrow \text{converges} \]

\[ \Rightarrow \text{converges} \]

**7701 Lecture #8**

1. Class do: \( \int 1 + e^{\beta x} dx \) with \( \beta \) real and \( \beta > \text{Re}(a) \)

2. Why are these relevant?

3. Questions to answer:
   - How to choose \( \alpha \)? Want \( \int \alpha \to \int \alpha \) \( e^{\beta x + i\alpha} \) \( 1 + e^{\beta x + i\alpha} \) \( \Rightarrow \frac{\alpha(x + \pi)}{1 + e^{\beta x + i\alpha}} \)
   - What is simplest choice for \( \beta \)?
   - Do the integrals on \( C_2 \) and \( C_4 \) vanish? Why?
   - As \( R \to \infty \), what was? (and why do the conditions matter?)
   - Where are the poles? Does \( e^{az} \) have singularities?
   - Where does \( 1 + e^{\beta z} \) have zeros for real \( \beta \)?

4. How do you tell Mathematica that \( \beta > \text{Re}(a) > 0 \)?
   - Assumptions \( \Rightarrow \beta > \text{Re}(a) > 0 \)
   - Use this in Integrate Function.
Flash back to (13) on dispersion relations. (Cahill 5.19)

Use of \( \frac{A}{x-x_0-i\epsilon} = \frac{1}{x-x_0} + \text{principal value} \)

\[ \text{note: } \text{when } (x-x_0) - i\epsilon \text{ has } -i\epsilon \]

\( \text{pole in upper half plane} \) \( \text{real functions} \)

Apply to a complex function \( f(z) = U(z) + iV(z) = \Re[f(z)] + i\Im[f(z)] \)

For which \( f(z) \) is analytic in the upper half \( z \) plane.

- Key example: dielectric constant of a material and \( f(z) \rightarrow 0 \) in the upper half plane.

Run for any \( Z_0 \) in the upper plane, \( z \to \infty \)

\[ \text{This contour as } R \to \infty, \text{ (real part vanishes)} \]

\[ \text{But now let } Z_0 = x_0 + i\epsilon \text{ with } x_0 \text{ real (certainly this is in upper half plane)} \]

\[ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(0)}{x-x_0-i\epsilon} \, dx = f(x_0) \] (because \( f(z) \) analytic so \( f(x+it) = f(x) \) \( \text{as } t \to 0 \))

\[ \therefore \frac{1}{2\pi i} \int_{-\infty}^{\infty} f(0) \, dx + \frac{1}{2\pi i} \int_{-\infty}^{\infty} f(x_0) \delta(x-x_0) \, dx = f(x_0) \]

\[ \therefore f(x_0) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(0)}{x-x_0} \, dx \]

\[ \therefore U(x_0) + iV(x_0) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{u(x)}{x-x_0} \, dx + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{v(x)}{x-x_0} \, dx \]

Separate real and imaginary: \( U(x_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{u(x)}{x-x_0} \, dx \) \( \text{real part is integral over} \)

\( V(x_0) = \frac{-i}{2\pi} \int_{-\infty}^{\infty} \frac{u(x)}{x-x_0} \, dx \) \( \text{imaginary part is integral over} \)
Comments on differential equation problems in 5#3:

Problems 2 and 3 are basic applications of the Frobenius method; we'll recap, but remember the initial equation is its starting point.

- one is a power series (you are only to find its regular solution).
- the other has power series times non-integer powers of x.
- you are expected to recognize common series; sum the series for functions like $e^x$, sin x, as x, etc.
- For Mathematica, see the "Solving some 5#3 differential equations" on the example notebook page.

Problem 4: $(1+x)y''+(3+2x)y'+(3+6x)y=0$

- look for $x>0$ solution.
- do (53) for asymptotic methods.
- what does "direct integration" mean?

eg. $V'=V$, let $w=V \Rightarrow \frac{dw}{w}=dx \Rightarrow \int \frac{dw}{w} = \ln w \Rightarrow w = e^{\ln w} = x$

$\Rightarrow \ln w = w(0) \Rightarrow w(x) = w(0) e^x$

$\Rightarrow V(x) = V(0) e^x \Rightarrow$ integrate again: $V(x) - V(0) = V(0) e^x$

Problem 5: $y''+(l+\frac{1}{4})y=0$ Take $y(0)=e^{-ax^2} x^a$ and find a choice of $a$ that gives a simpler equation for $V$ (i.e., no $x^4$).

In general: extract the dominant part of the solution and then solve the rest approximately (or numerically).

Convergence of power series solutions to differential equations: out to singular points. Check Legendre: $y'' - \frac{2x}{1-x^2} y' + \frac{\Delta_l(x+a)}{1-x^2} y = 0$

Ratio test for power series at given $x$: $\lim_{n \to \infty} \frac{|a_n (x-x_0)|}{|a_{n+1}|}$

Legendre: $\frac{\Delta_{l+2}}{\Delta_l} = \left( \frac{j+1}{(j+2)} \right) \frac{a_{l+2}}{a_j} \Rightarrow |(1/(j+1)_{j+1} x(x+a)^2)| \frac{a_{l+2}}{(j+2)(j+2)/j} \Rightarrow 1 \Rightarrow |x|<1$
9/9/13

**Reply: Frobenius method and the indicial equation**

- **General results for series solutions.**

1. **If** $x = x_0$ **is an ordinary point** of
   \[ y''(x) + P(x)y'(x) + Q(x)y(x) = 0 \]
   (so $P(x)$ and $Q(x)$ are analytic at $x_0$), then the general solution is
   \[ y(x) = C_1 \left( x - x_0 \right)^a + C_2 \left( x - x_0 \right)^b \]
   linearly independent (what does that mean?)

   - The radius of convergence of each series is at least as great as the distance from $x_0$ to the nearest singular point of the differential equation.

2. If we have singular points that are irregular, e.g., $x_0$ and $x_0$ diverge at $x_0$, and $\lim_{x \to x_0} x^k (x - x_0)^n$ respectively, then life is complicated. E.g., on $PS \# 4 \Rightarrow$ essential singularity in solution, uncommon in equations for physical systems.

3. If $x = 0$ is a regular singular point of
   \[ y''(x) + p(x)y'(x) + q(x)y(x) = 0 \]
   and $p_1$ and $p_2$ are the roots of the indicial equation with $p_1 \neq p_2$, then
   \[ y_1(x) = x^{p_1} \sum_{n=0}^{\infty} c_n x^n \quad \text{and} \quad y_2(x) = x^{p_2} \sum_{n=0}^{\infty} b_n x^n \]
   \[ (a_1 + b_1) = 0 \]

4. If $p \neq q$, then
   \[ y_1(x) = x^{p_1} \sum_{n=0}^{\infty} c_n x^n \quad \text{and} \quad y_2(x) = x^{p_2} \sum_{n=0}^{\infty} b_n x^n \]
   where $c$ can be zero.

5. Do not use $PS$ and lowest equation in $x$; (fig.) $a_1 = 0$ to find more $a$ values. Use it with $p_1$ and $p_2$ to decide if $a_1 = 0$.
   E.g., if $p_1 = 0$, then $a_1 = 0$ is required; if $p_2 = 0$, then $a_2$ can be anything.
9/9/13

The Gamma Function \( \Gamma(x) \) — [see Chap. 6 in Arfken, Sect. 2.9 in Lea]

In the solution to Laguerre's equation in problem 2 of Arfken, you find the series solution:

\[
y(x) = a_0 \sum_{n=0}^\infty \frac{(-1)^n a(x-1) \cdots (x-(n-1))}{(n!)^2} x^n
\]

How could we write the \( n \)th term more generally?

Consider \( x = k \), an integer,

\[
\Rightarrow 1 \cdot (k-1) \cdots (k-(n-1)) = k(k-1) \cdots (k-(n-1))(k-n)(k-n-1) \cdots 3 \cdot 2 \cdot 1 \over (k-n)(k-(n+1)) \cdots 3 \cdot 2 \cdot 1
\]

\[
= \frac{k!}{(k-n)!}
\]

Can we generalize? Yes, with \( \Gamma(z+1) = \Gamma(z) \), we have \( \Gamma(x+1) \) is equal to \( x(x-1)(x-(n-1)) \) part.

The Gamma function is defined as

\[
\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, dt \quad \text{for } x > 0 \ (\text{real})
\]

\[
\Gamma(n) = \int_0^\infty e^{-t} t^{n-1} \, dt = -e^{-t} \bigg|_0^n = 1
\]

We can integrate by parts to show that \( \Gamma(x+1) = x \Gamma(x) = x(x-1) \Gamma(x-1) \)

\[
\Gamma(n+1) = n!
\]

\[
\text{eg. } u = e^{-t} \Rightarrow du = -e^{-t} \, dt, \quad dv = t^{x-1} \, dt \Rightarrow v = \frac{t^x}{x}
\]

\[
\Rightarrow \Gamma(x) = \frac{e^{-t} t^x}{x} \bigg|_0^\infty + \int \frac{t^x}{x} e^{-t} \, dt = \frac{1}{x} \Gamma[x+1]
\]
\[ \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-t^2} dt = 2 \int_0^\infty e^{-u^2} \frac{du}{\sqrt{u}} = \int_0^\frac{\pi}{2} e^{-\cos^2 \theta} \sin \theta d\theta = \frac{\sqrt{\pi}}{2} \]

Claim: \( \int e^{x^2} dx = \frac{1}{2} \Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{2} \right) = \frac{\sqrt{\pi}}{2} \) [switch to \( u = x^2 \)] and \( \sqrt{\frac{x}{\pi}} e^{-x/2} dx = \left( \frac{1}{x} \right) \Gamma \left( \frac{1}{2} \right) = \frac{\sqrt{\pi}}{2} \).

How do we define the integral for \( \Re(z) > 0 \)? Use a contour integral.

Let \( \gamma = \gamma_1 \gamma_2 \gamma_3 \), use this contour so that we include the top of the branch backwards and the bottom from \( \Re(z) = 0 \) to \( \infty \), with \( \theta = \pi \).

The little circle around the branch point vanishes:

\[ \int e^{-t} e^{t \cdot \gamma} dt = \int_0^{2\pi} \left( e^{i \theta} \right)^{z-1} e^{-i \theta} e^{i \theta} dt = \int_0^{2\pi} e^{i \theta} e^{i \theta} \left( e^{i \theta} \right)^{z-1} e^{-i \theta} \]

\[ = \int_0^{2\pi} e^{i \theta} e^{i \theta} \left( e^{i \theta} \right)^{z-1} e^{-i \theta} = 0 \text{ for } \Re(z) > 0 \]

On the bottom of the branch, we get an extra \( e^{i \pi z} \), so

\[ \gamma(z) = \frac{1}{e^{i \pi z} - 1} \int \frac{e^{-t} e^{t \cdot \gamma}}{e^{2\pi}} \]

when \( z \) is positive integer, defined as limit. There are simple poles for \( z \) non-negative integer.

Ok, but what about \( \left( \gamma - \frac{1}{2} \right) \)? Define by analytic continuation using \( \gamma(z) = \gamma(z+1)/z \) sufficiently many times. E.g. \( (\gamma - \frac{1}{2}) = \gamma \left( \frac{1}{2} \right)/\frac{1}{2} \)

(repeat until \( \Re(z+1) > 0 \))