office hours later Monday afternoon

6/26/13

7701 Lecture 3

Reminder: PS#1 due in Furnstahl mailbox in main office before the colloquium on Tuesday. Please attend colloquium when possible.

Questions on Sin x - Eijk problems?

Warm-ups:
(a) Spot the error: \((\sin z)^2 = \frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{iz}}{2} = \sin z \)?

must be wrong because it implies \( \sin z \) is real.
try small \( z \) to check: \((3)^2 = 9 \neq 3 \) only if \( z \) is real
problem: \( z \to \overline{z} \) in \((\sin z)^2 \) \( \Rightarrow \) answer is \( z \star \)

(b) What is \( \ln |z| ? \) (ans: \( \sin x + \sinh y \))

(c) What are the zeros of \( \sin z ? \) (ans: \( x = 0, \pm \pi , \pm 2\pi \) and \( y=0 \))

(d) Without tables, sketch \( \cosh y \) and \( \sinh y \) based on \( y \to \infty \)

Mathematica: small \( y \) Series \([ \cosh y, \{ y, 0, 3 \}] \Rightarrow 1 + \frac{y^2}{2} + O[y]^4 \)

large \( y \) Series \([ \cosh y, \{ y, \infty \}] \Rightarrow \cosh y \), doesn't work.

Lecture plan: * Go back to [3], then do [15]-[17] * Continue with [19]+ as time permits
Analyticity and Taylor expansions

- If a function is differentiable at $z = z_0$, then its Taylor expansion exists:
  \[ f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \]

- Existence of Taylor series at $z_0 \iff$ analytic at $z_0$ (alternative definition)

- Series "works" (that is, it converges) in circular region about $z_0$ up to first singularity (pole, branch point, essential singularity).

  \[ \text{This is } \infty \text{ for } e^z, \sin z, \text{ such as } \frac{1}{z^2}, \text{ etc.} \]

  \[ \text{For } \frac{1}{z^2}, \text{ about } z = 0, \text{ never vanishes for } x \text{ real: } 1 - x^3 + x^4 - x^5, \]
  \[ \text{but only converges for } |z| < 1 \text{ because } \frac{1}{z^2} = \frac{1}{z^2} \]

  \[ \text{Mathematica notebooks complex series ab (later)} \]

- Generalization: Laurent expansion a series:
  \[ f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n \]

  \[ \text{For example, } f(z) = \frac{1}{z^2 - 1} = \frac{1}{z - 1} + \frac{1}{z + 1} + O(z) \]

  \[ \Rightarrow a_n = 0 \text{ for } n < 0 \text{ except } a_{-1} = 1, \quad n = -1 \text{ coefficient is the residue} \]

  \[ \text{(much more to come!)} \]

- Mathematica gives you Laurent expansion using Series[]

- We can find the Laurent expansion in simple cases, e.g., analytic functions times explicit poles.

  Plan: expand functions in Taylor series (about specified point!)

  and then combine the two parts term by term.

- In problem set: $\frac{\cos z}{z - 1}$ about $z = 1 \neq \frac{1 - \frac{z^2}{2} + \frac{z^4}{24} + \cdots}{z - 1}$

  \[ \frac{z}{z - 1} \quad \text{about } z = 1 \quad \text{"Spot the Error!"} \]

  What are the singularities for each? (Class question)
Example: expand \( f(z) = \frac{e^z}{(z-2)^3} \) about \( z = 2 \)

Spot the Error! \( \frac{1}{(z-2)^3} = \frac{1}{(z-2)^3} + \frac{1}{(z-2)^2} + \frac{1}{(z-2)} + \cdots \) expanded about \( z \)\!

Plan: write \( w = z - 2 \) and expand about \( w = 0 \) (which we often just write down)

\[ e^z = e^{w+2} = e^w e^2 = e^w (1 + w \frac{w^2}{2!} + \cdots) = e^2 (1 + \frac{w^2}{2!} + \frac{w^4}{4!} + \cdots) \]

\[ v = \frac{1}{m!} \]

\[ f(z) = e^2 \left( \frac{1}{(z-2)^3} + \frac{1}{(z-2)^2} + \frac{1}{(z-2)} + \cdots \right) \]

\[ = \sum_{n=-\infty}^{\infty} a_n (z-2)^n \quad \text{where} \quad a_n = \left\{ \begin{array}{ll} e^2 (n+3)! & \text{if } n > -3 \\ 0 & \text{if } n \leq -3 \end{array} \right. \]

What is the residue? Coefficient of \( \frac{1}{z-2} \Rightarrow e^2/2 \)

Why is the \( n = -1 \) term special? (We'll see.)

Two other ways to find the residue

- Mathematica Residue \( \lim_{z \to 2} \left( (z-2)^3 f(z) \right) = 2, 2^3 \)
- \( m\)th order pole \( \left( m = 3 \text{ here} \right) \)

\[ \text{Res} (a) = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [ (z-a)^m f(z) ] = \frac{e^2}{2} \frac{d^2}{dz^2} e^z \bigg|_{z=2} = \frac{e^2}{2} \]

Region of convergence?

\[ e^z \text{ converges everywhere but we have to exclude } \]

At point \( z = 2 \Rightarrow |z-2| > 0 \) is region of convergence

\[ \int_{C} f(z) \, dz = \text{sum of residues at } \text{enclosed poles} = \text{sum of residues at } \{ z_i \text{, } z_i \} \]

\[ \text{Residue:} \]

\[ \sum_{z_i} \text{residuals of enclosed poles} \]
Summary of Convergence of Taylor and Laurent series:

- A Taylor series about \( z = z_0 \) has a convergent expansion in a circle out to the first non-analytic point.
  - pole \( z = z_0 \)
  - branch point

- Function is defined by Taylor expansion inside "radius of convergence" (radius of circles in figures).
  - converges inside, diverges outside
  - illustrate by Mathematica complex series notebook for one or two examples (try just inside and just outside)
  - extend definition further by analytic continuation (more in texts and later)

- Try Laurent series in Mathematica as well.
  - Derivations of Laurent series are in my texts. Key generalization is they are valid in an annular region, rather than filled-in circle.

Singularities: Vocabulary

- Isolated singular point if not analytic only at \( z = z_0 \)
  - \( f(z) = \frac{a_n}{z-z_0} + a(z-z_0)^2 + \ldots \Rightarrow \) simple pole with residue \( a_n \)
  - if only isolated poles, \( f(z) \) is "meromorphic"
  - order \( m \) pole if \( (z-z_0)^m f(z) \) is not singular at \( z_0 \) but non-zero.
  - essential singularity if all \( n \to \infty \) contributions, eg. \( e^{1/z} \)
  - branch point: multivalued \( f(z) \) when circling branch point \( z_0 \)
Integrals in the Complex Plane.

In analogy to the "Riemann sum" definition of a integral \( \int f(x) \, dx = \lim_{N \to \infty} \sum_{i=1}^{N} f(x_i) \Delta x_i \), we define

\[
\int_{C} f(z) \, dz = \lim_{n \to \infty} \sum_{j=1}^{n} f(z_j) \frac{(z-j-1)}{\Delta z_j}
\]

where the limit should be independent of how \( \Delta z_j \)'s are spaced or what intermediate \( \Delta z_j \)'s are chosen.

An alternative definition as a line integral uses \( f = u + iv \):

\[
\int_{C} f(z) \, dz = \int_{C} (u(x,y) + iv(x,y)) \, dx - dy
\]

\[
= \int_{C} u(x,y) \, dx - v(x,y) \, dy + i \int_{C} v(x,y) \, dx + u(x,y) \, dy
\]

Consider \( f(z) = z^n \), \( n \) integer and contour \( C \) a circle of radius \( r \), traversed counterclockwise (increasing \( \theta \)).

First \( n \neq -1 \):

\[
\int_{C} z^n \, dz = \int_{0}^{2\pi} r^n e^{in\theta} (ire^{i\theta} \, d\theta) = \int_{0}^{2\pi} r^{n+1} e^{in\theta} \, d\theta
\]

\[
= \frac{r^{n+1}}{in} e^{in\theta} \bigg|_{0}^{2\pi} = \frac{r^{n+1}}{in} (1 - e^{2\pi in}) = \frac{r^{n+1}}{in} (0) = 0
\]

For \( n = -1 \):

\[
\int_{C} \frac{1}{z} \, dz = \int_{0}^{2\pi} e^{i\theta} \, d\theta = 2\pi i
\]

Note that \( n = -2, -3, \ldots \) are zero; just \( n = -1 \) is different.

If \( f(z) = (z-z_0)^n \), replace \( z \to z_0 + re^{i\theta} \) to get the result and radius \( r \) about \( z_0 \), \( k \) constant.
Now consider a different path around a rectangle.

\[ \int_C z^n \, dz = \int_a^b (a+iy)^n \, dy + \int_{a+ib}^{b+ia} (x+ib)^n \, dx + \int_{b+ia}^{-a} (-a+iy)^n \, dy + \int_{-a}^a (x-iy)^n \, dx \]

If \( n \neq -1 \):

\[ \int_C z^n \, dz = \frac{(a+iy)^{n+1}}{n+1} \bigg|_a^b + \frac{(x+ib)^{n+1}}{n+1} \bigg|_{a+ib}^{b+ia} + \frac{(-a+iy)^{n+1}}{n+1} \bigg|_{b+ia}^{-a} + \frac{(x-iy)^{n+1}}{n+1} \bigg|_{-a}^a \]

\[ = \left[ \frac{(a+ib)^{n+1}}{n+1} - \frac{(a+ib)^{n+1}}{n+1} \right] - 0 = 0 \quad \text{(again!)} \]

Quicker:

\[ \int_C z^n \, dz = \frac{z_{a+ib}^{n+1}}{n+1} \bigg|_{a+ib}^{b+ia} + \frac{z_{b+ia}^{n+1}}{n+1} \bigg|_{b+ia}^{-a} + \frac{z_{-a}^{n+1}}{n+1} \bigg|_{-a}^a = 0 \quad \text{(cancel in pairs again)} \]

What if \( n = -1 \)?

\[ \int_C z^{-1} \, dz = \ln (a+iy) \bigg|_a^b + \ln (x+ib) \bigg|_a^b + \ln (-a+iy) \bigg|_b^{-a} + \ln (x-iy) \bigg|_{-a}^a = 0 \]

Looks like canceling in pairs to get zero again, but we need to be more careful!

\[ \ln z = \ln r + i\theta \Rightarrow \text{the log parts will cancel} \]

but \( G \) increases by \( \pi i i \) in going from \( z_1 \) back to \( z_1 \)

\[ \Rightarrow \text{Integral gives } i \pi \]

\[ \Rightarrow \text{imaginary parts of } \ln z \]

**Bold extrapolation:**

- We always get \( \int_C z^n \, dz = \begin{cases} 0 & \text{if } n = -1 \\ 0 & \text{if } n = -1 \text{ and } C \text{ is counterclockwise} \end{cases} \) for any \( C \) around \( 0 \)

- Turns out to be true for \( n = -1 \) generally and for \( n = -1 \) if the point \( z = 0 \) is inside \( C \).
More generally we have Cauchy's Theorem:

\[ \oint_C f(z) \, dz = 0 \quad \text{if } f(z) \text{ is analytic in a simply connected region } \Omega \text{ that includes } C. \]

This result is useful for evaluating contour integrals in the complex plane, where the integral is independent of the path taken around the closed loop.

Analytic in \( \Omega \) means the Cauchy-Riemann equations hold inside \( \Omega \). These equations involve derivatives in the interior of a region, so they should tell us about what happens on an integral around the boundary: \( \nabla \) leads to Stokes' theorem.

*Integral of derivatives in interior is equal to function on boundary (we'll come back to \( \nabla \) later in the semester).

Let's see that this is just Stokes's theorem in xy plane. Suppose \( \mathbf{A} = \hat{x} A_x + \hat{y} A_y + [\hat{z} 0] \). Choose \( \mathbf{A} \) and \( \mathbf{S} \) to match:

\[ \oint_C \mathbf{A} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{A}) \cdot dS = \iint_S \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \, dx \, dy \]

Recall \( \oint_C f(z) \, dz = \oint_C (udx - vdy) + i \oint_C (vdx +udy) \)

\[ \oint_C (udx - vdy) = \iint \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \, dx \, dy = 0 \]

\[ \oint_C (vdx +udy) = \iint \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \, dx \, dy = 0 \]

* More complete proof given in Aftken. See also O'Hallan.
What about have $\frac{1}{z}$ or, more generally, $\frac{1}{z - z_0}$?

**Cauchy's Integral Formula:**

\[ \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(z)}{z - z_0} \, dz = g(z_0) \]  

where $z_0$ is any point in the region bounded by $\mathcal{C}$, while $g(z)$ is analytic on and in $\mathcal{C}$.

General construction to see why this works: (shows we can deform contours)

By adding extra pieces to our desired contour $\mathcal{C}$ (the straight line in and back to $z_0$ and integral $\mathcal{C}_2$ in a circle around $z_0$), we can now apply Cauchy:

\[ \oint_{\mathcal{C}} \frac{g(z)}{z - z_0} \, dz + \oint_{\mathcal{C}_2} \frac{g(z)}{z - z_0} \, dz = 0 \]

But \[ \oint_{\mathcal{C}_2} \frac{g(z)}{z - z_0} \, dz \xrightarrow{\text{by changing variables}} \int_0^{2\pi} g(z_0 + re^{i\theta}) \, r i e^{i\theta} \, d\theta \]

\[ = i \int_0^{2\pi} g(z_0 + re^{i\theta}) \, r \, d\theta \xrightarrow{\text{as } r \to 0} \lim_{r \to 0} ig(z_0) \int_0^{2\pi} \, d\theta = \pi i g(z_0) \]

\[ \Rightarrow \oint_{\mathcal{C}} \frac{g(z)}{z - z_0} \, dz = 2\pi i g(z_0) \] QED

Once you know values on the boundary of an analytic region, you know them in the interior.

Generalize: \[ f(z_0) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(z)}{z - z_0} \, dz \]

\[ f''(z_0) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(z)}{(z - z_0)^3} \, dz \]

\[ f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{\mathcal{C}} \frac{f(z)}{(z - z_0)^{n+1}} \, dz \]
Apply Cauchy's formula to simple case:

\[ \oint_{c} \frac{g(z)}{(z-z_0)^n} \, dz = \begin{cases} 0 & \text{if } n \neq -1 \text{ because analytic} \\ 2\pi i & \text{if } n = -1 \text{ take } g(z) = 1 \end{cases} \]

so if we have a function with a Taylor series, then term by term \( f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n \) for \( n > 0 \) terms vanish, and \( \oint f(z) \, dz = 0 \Rightarrow \text{Cauchy's theorem} \).

\[ f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n \Rightarrow \text{analytic function} \]

\( \Rightarrow \) works up to first non-analytic point in \( \mathbb{D} \) region.

\( \Rightarrow \) determines "radius of convergence".

- this is \( \frac{1}{2} \) for \( e^z, \sin z \)
- for \( \frac{1}{z+1} \) about \( z=0 \), \( |z| < 1 \) because \( \frac{1}{z+1} \Rightarrow \text{even though never vanishes for } z \in \mathbb{R}, \text{doesn't converge for } |z| = 1 \).

\( \Rightarrow \) check with Mathematica (see notebook complex_series.nb)

If we have poles, as before we generalize with a Laurent series or expansion:

\[ f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \quad \text{where} \quad a_n = \frac{1}{2\pi i} \oint_{\gamma_n} \frac{f(z) \, dz}{(z-z_0)^{n+1}} \]

- eg, \( f(z) = (e^z-1)^{-1} = \frac{1}{z} - \frac{1}{2} + \frac{z}{12} + O(z^3) \) in annular region about \( z=0 \)

So now term-by-term will give zero except \( \Re \text{e}^{a_\frac{1}{2}} \frac{1}{x \sqrt{1}} \)

\( \Rightarrow a_{-1} \text{ is the residue} \).
This is all we need for the residue theorem

\[ \oint_C \frac{f(z)}{z - z_i} \, dz = 2\pi i \left( a_{-1,z_i} + a_{+z_i} \right) \]

= \text{sum of "residues" of enclosed poles}

Here \( f(z) = \sum_{n=-\infty}^{\infty} \frac{a_n}{z - z_i} \) about \( z_i \), \( a_{\pm z_i} \) is the coefficient.

Apply this to calculate many definite integrals that arise in mathematical physics.

Aside: List of methods for definite integrals from Arfken:
1. contour integration
2. convert to gamma or beta function
3. numerical quadrature
4. integral transforms
5. series expansion and term-by-term integration

Mathematica might use any of these methods (at your request or automatically).

Let's steps for evaluating an integral \( \int_a^b f(x) \, dx \) as simple example:
1. Draw complex z plane with contour C
   - chosen to include integral of interest
   - pole poles or other singularities, including branch cuts.
2. If there is a branch cut, "deform" the contour so it doesn't hit the cut.
3. Note poles inside C (here \( z = \pm i \))
4. Evaluate the residue of \( f \) at each enclosed pole:

   \[ \frac{1}{z+i} = \frac{1}{i} \left( \frac{1}{z} \right) = \frac{1}{i} \frac{1}{z} \]

5. Apply the residue theorem \( \oint_C \frac{f(z)}{z - z_i} \, dz = 2\pi i \left( a_{-1,z_i} + a_{+z_i} \right) \)

6. Evaluate other non-vanishing parts (eg. integrals on both sides of branch cut)
Ways to find a residue

1. Use Mathematica: Residue[f[z], {z, a}]

2. Find the Laurent series and pick out the $a^{-1}$ coefficient, often with a simple Taylor expansion of non-pole piece.

3. Pole at $a$:
   \[ \text{Res } f(z) = \lim_{z \to a} (z-a) f(z) \]

   - most common rule
   \[ f(z) = \frac{1}{z-a}, \quad a = 1 \Rightarrow \text{Res } f(z) = \lim_{z \to 1} (z-1) \frac{1}{2z+1} \]

   - If $f(z) = \frac{\sin z}{2z+1} \Rightarrow \text{Res } f(z) = \frac{\sin 1}{2} = \frac{1}{2i}$

4. Pole of order $m$:
   \[ \text{Res } f(z) = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z-a)^m f(z)] \]

   - Example: $f(z) = \frac{e^z}{(z-1)^2} \Rightarrow \text{Res } f(z) = \lim_{z \to 1} \frac{1}{(z-1)^2} \frac{d}{dz} [(z-1)^2 e^z] = \frac{e}{2}$

5. Evaluate integrals

\[ \frac{1}{2\pi i} \int_C f(z) \, dz \]

- Max examples from integrals later