Spin-1/2 dynamics

The intrinsic angular momentum of a spin-1/2 particle such as an electron, proton, or neutron assumes values $\pm \hbar / 2$ along any axis. The spin state of an electron (suppressing the spatial wave function) can be described by an abstract vector or ket a concrete realization of which is a two-component column vector.

The intrinsic angular momentum of a particle is a vector operator whose components obey the standard angular momentum commutation relations. Since a spin-1/2 particle has two possible results of a measurement they can be described by $2 \times 2$ matrices. Recall the Pauli representation:

$$\vec{S} = \frac{\hbar}{2} \vec{\sigma}$$

where $\vec{\sigma}$ are the Pauli spin matrices defined by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  \hspace{1cm} (2)

The choice makes $S_z$ diagonal. Recall the logic of how these are determined. We can write down $S_z$ since it is diagonal and the diagonal elements are the eigenvalues. The eigenvectors are given by

$$|z+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } |z-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$  \hspace{1cm} (3)

We use the definitions of $S_+$ and $S_-:

$$S_+ \langle z+ | = 0 \text{ and } S_+ \langle z- | = | z+ \rangle$$

$$S_- \langle z+ | = | z- \rangle \text{ and } S_- \langle z- | = 0$$

allow us to find that

$$S_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } S_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$ (Knowing $S_+$ we can find $S_-$ since it is the Hermitian conjugate of $S_+$.) Since $S_\pm = S_x \pm i S_y$ we can find $S_x$ and $S_y$.

Some simple properties that you should verify and learn to use:

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I.$$  \hspace{1cm} (1)

$$\sigma_x \sigma_y = i \sigma_z, \quad \sigma_y \sigma_x = i \sigma_z, \quad \sigma_x \sigma_z = \sigma_z \sigma_x = i \sigma_y.$$  \hspace{1cm} (2)

So the “standard” basis corresponds to spin up and down along the z-axis. In particular if the particle is in a state described by the ket

$$|s\rangle \rightarrow \begin{pmatrix} a \\ b \end{pmatrix}$$  \hspace{1cm} (3)
then the probability of finding $+\hbar/2$ upon making a measurement of the spin along the $z-$axis is simply $a^*a = |a|^2$. Absolutely explicitly this probability is given by the squared absolute value of the “overlap” matrix element

$$\langle z | s \rangle = (1, 0) \begin{pmatrix} a \\ b \end{pmatrix} = a .$$  (4)

**Problem:** Consider a spin-1/2 particle with spin pointing up along $\hat{n}$ given by

$$\hat{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) .$$  (5)

What is the 2-component vector (called a *spinor*) that corresponds to this state?

It is important to note that we are measuring the spin along $\hat{n}$ and the operator corresponding to this observable is $\vec{S} \cdot \hat{n}$. It is given by

$$\vec{S} \cdot \hat{n} = \frac{\hbar}{2} (\sin \theta \cos \phi \sigma_x + \sin \theta \sin \phi \sigma_y + \cos \theta \sigma_z)$$  (6)

$$\vec{S} \cdot \hat{n} = \frac{\hbar}{2} \left[ \begin{array}{cc} 0 & \sin \theta \cos \phi \\ \sin \theta \cos \phi & 0 \end{array} \right] + \left[ \begin{array}{cc} 0 & -i \sin \theta \sin \phi \\ i \sin \theta \sin \phi & 0 \end{array} \right] + \left[ \begin{array}{cc} \cos \theta & 0 \\ 0 & - \cos \theta \end{array} \right]$$

$$\Rightarrow \vec{S} \cdot \hat{n} = \frac{\hbar}{2} \begin{pmatrix} \cos \theta + \sin \theta e^{-i\phi} \\ \sin \theta e^{i\phi} - \cos \theta \end{pmatrix} .$$  (7)

We need to find the eigenvalues and eigenvectors. Consider the matrix without the factor of $+\hbar/2$. Note that the trace defined to be the sum of the diagonal matrix elements. The trace is also the sum of the eigenvalues; denoting them by $\lambda_1$ and $\lambda_2$ we have $\lambda_1 + \lambda_2 = 0$. The determinant is easily calculated to be $-1$ and this is the product of the eigenvalues. Thus we find $\lambda_1 \lambda_2 = -1$. Together we have $\lambda_1 = 1$ and $\lambda_2 = -1$. Thus the eigenvalues of $\vec{S} \cdot \hat{n}$ are $\pm \hbar/2$. This shows the result that the spin measured along any arbitrary axis yields only two possible values, $\pm \hbar/2$. This is an amazing feature of quantum mechanics. Please spend a few minutes thinking about what happens classically.

Let us denote the eigenvectors by $|\hat{n}+\rangle$ and $|\hat{n}-\rangle$. We can determine these easily.\(^1\)

$$|\hat{n}+\rangle = \begin{pmatrix} \cos \frac{\theta}{2} e^{-i\phi} \\ \sin \frac{\theta}{2} \end{pmatrix} \quad \text{and} \quad |\hat{n}-\rangle = \begin{pmatrix} \sin \frac{\theta}{2} e^{-i\phi} \\ -\cos \frac{\theta}{2} \end{pmatrix} .$$  (8)

\(^1\)We have for example

$$\frac{\hbar}{2} \begin{pmatrix} \cos \theta & \sin \theta e^{-i\phi} \\ \sin \theta e^{i\phi} & -\cos \theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} a \\ b \end{pmatrix} .$$

Thus we have (canceling $\hbar/2$)

$$\cos \theta a + \sin \theta e^{-i\phi} b = a \Rightarrow \frac{a}{b} = \frac{\sin \theta}{1 - \cos \theta} e^{-i\phi} = \frac{2 \sin(\theta/2) \cos(\theta/2)}{2 \sin^2(\theta/2)} e^{-i\phi} = \frac{\cos(\theta/2)}{\sin(\theta/2)} e^{-i\phi}$$

where we have used the half-angle formulae. We have chosen $a = \cos(\theta/2)e^{-i\phi}$ and $b = \sin(\theta/2)$. 

2
This is the solution to Problem 4.31 in Griffiths (page 160) except for an overall phase factor of \( \exp(i\phi) \). Feynman in Vol. III gives a more symmetrical formula by multiplying by \( e^{i\phi/2} \) (Equation 10.30):

\[
|\hat{n}+\rangle = \begin{pmatrix} \cos \frac{\theta}{2} e^{-i\phi/2} \\ \sin \frac{\theta}{2} e^{i\phi/2} \end{pmatrix} \quad \text{and} \quad |\hat{n}-\rangle = \begin{pmatrix} \sin \frac{\theta}{2} e^{-i\phi/2} \\ -\cos \frac{\theta}{2} e^{i\phi/2} \end{pmatrix}
\]  

Problem: Given a spin in the state \(|z+\rangle\), i.e., pointing up along the z-axis what are the probabilities of measuring \(\pm \hbar/2\) along \(\hat{n}\)?

The probability of measuring up is given by \(|\langle \hat{n}+ |z+\rangle|^2\). This is

\[
\left| \begin{pmatrix} \cos(\theta/2)e^{i\phi} \\ \sin(\theta/2)(1/0) \end{pmatrix} \right|^2 = |\cos(\theta/2)e^{i\phi}|^2 = \cos^2(\theta/2).
\]

The probability of measuring \(-\hbar/2\) along \(\hat{n}\) given that the spin points up along \(z\) is \(\sin^2(\theta/2)\). Please verify this explicitly.

How does one interpret this classically? Classically the angular momentum along \(\hat{n}\) is \((\hbar/2)\cos \theta\). We have to compare the classical result with the quantum mechanical expectation value \(\langle \vec{S} \cdot \hat{n} \rangle\). The expectation value or the mean values is given by the sum of the to possible values \(\pm \hbar/2\) multiplied by their corresponding probabilities:

\[
\frac{\hbar}{2} \cos^2(\theta/2) + \left( -\frac{\hbar}{2} \right) \sin^2(\theta/2) = \frac{\hbar}{2} \left( \cos^2(\theta/2) - \sin^2(\theta/2) \right) = \frac{\hbar}{2} \cos \theta.
\]

This is an example of how expectation values conform to classical expectations in this the most quantum mechanical of systems.
Consider the dynamical problem of a spin-1/2 particle in a magnetic field. We will study the simplest case by choosing $\hat{z}$ along the magnetic field. We study the problem classically first. In a uniform magnetic field the magnetic moment $\vec{\mu}$ experiences a torque $\vec{\mu} \times \vec{B}$. Since the magnetic moment is proportional to the angular momentum we have $\vec{\mu} = \gamma \vec{J}$. Recall that the rate of change of the angular momentum is the torque:

$$\frac{d\vec{J}}{dt} = \vec{\mu} \times \vec{B} = \gamma \vec{J} \times \vec{B}.$$  \hspace{1cm} (10)

Choosing $\vec{B} = B_0 \hat{z}$ and defining $\omega_L = \gamma B_0$ we can write down the equations for each component:

$$\dot{J}_x = \omega_L J_y, \quad \dot{J}_y = -\omega_L J_x, \quad \text{and} \quad \dot{J}_z = 0.$$

Clearly $J_z$ the projection along the magnetic field is a constant in time. We solve the other two equations by a useful trick. Multiplying the equation for $J_y$ by $i$ and adding to the $J_x$ equation we have

$$\dot{J}_x + i\dot{J}_y = \omega_L (J_y - iJ_x) = -i\omega_L (J_x + iJ_y).$$

Recall that $\dot{f} = -i\omega_L f$ is easily solved as $f(t) = f(0) e^{-i\omega_L t}$. Check that this obeys the equation and the initial condition at $t = 0$. Thus we obtain

$$\dot{J}_x(t) + i\dot{J}_y(t) = (J_x(0) + iJ_y(0)) e^{-i\omega_L t}.$$

Let us choose $^2 J_y(0) = 0$ so that the moment is oriented in the $xz$-plane initially. We have

$$J_x(t) + iJ_y(t) = J_x(0) e^{-i\omega_L t} \Rightarrow J_x(t) = J_x(0) \cos(\omega_L t) \quad \text{and} \quad J_y(t) = -J_x(0) \sin(\omega_L t).$$

Thus we have the magnetic moment vector describing a cone with its tip moving in a circle with frequency $\omega_L$. This is referred to as Larmor precession.

Quantum mechanically we start with the hamiltonian for a magnetic field along the $z$-axis

$$H = -\gamma B_0 S_z = \omega_L \sigma_z = -\frac{\hbar}{2} \sigma_z = \begin{pmatrix} -\frac{\hbar}{2} \omega_L \sigma_z & 0 \\ 0 & \frac{\hbar}{2} \omega_L \end{pmatrix}.$$

We can compare with the classical result by evaluating expectation values as a function of time, for example,

$$\langle S_x \rangle(t) = \langle \chi(t)|S_x|\chi(t) \rangle.$$

$^2$The general case is easily solved by choosing

$$J_x(0) + iJ_y(0) = J_{\perp}(0) e^{i\phi}$$

to find

$$J_x(t) = J_{\perp}(0) \cos(\omega_L t - \phi) \quad \text{and} \quad J_y(t) = -J_{\perp}(0) \sin(\omega_L t - \phi).$$
We wish to find $|\chi(t)\rangle$. Since we are given that the spin points up along $\hat{n}$ in the $x - z$ plane we have $\hat{n} = (\sin \theta, 0, \cos \theta)$. We have computed the eigenvectors of $\vec{S} \cdot \hat{n}$ and therefore,

$$|\chi(t - 0)\rangle = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix}.$$  

We know that the eigenvalues of $H$ are $\pm \hbar \omega_L/2$ since it is diagonal. The corresponding eigenvectors are

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$  

Expanding the initial state in terms of the eigenfunctions of $H$ we have

$$|\chi(t - 0)\rangle = \cos(\theta/2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin(\theta/2) \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$  

Therefore, at time $t$ we have

$$|\chi(t)\rangle = e^{-i\omega_L t/2} \cos(\theta/2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{i\omega_L t/2} \sin(\theta/2) \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$  

We find

$$|\chi(t)\rangle = \begin{pmatrix} e^{-i\omega_L t/2 \cos(\theta/2)} \\ e^{i\omega_L t/2 \sin(\theta/2)} \end{pmatrix}.$$  

Therefore, we have

$$\langle \chi(t)|S_x|\chi(t)\rangle = \frac{\hbar}{2} \begin{pmatrix} e^{i\omega_L t/2 \cos(\theta/2)} \\ e^{i\omega_L t/2 \sin(\theta/2)} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\omega_L t/2 \cos(\theta/2)} \\ e^{i\omega_L t/2 \sin(\theta/2)} \end{pmatrix}$$

$$= \frac{\hbar}{2} \left( e^{i\omega_L t/2 \cos(\theta/2)} + e^{-i\omega_L t/2 \cos(\theta/2)} \right) \left( e^{i\omega_L t/2 \sin(\theta/2)} + e^{-i\omega_L t/2 \sin(\theta/2)} \right)$$

$$= \frac{\hbar}{2} \sin(\theta/2) \cos(\theta/2) \left( e^{i\omega_L t} + e^{-i\omega_L t} \right)$$

$$= \frac{\hbar}{2} \sin(\theta/2) \cos(\theta/2) \times 2 \cos(\omega_L t) = \frac{\hbar}{2} \sin \theta \cos(\omega_L t).$$  

Note that $\langle S_x \rangle(t = 0)$ is precisely $(\hbar/2) \sin \theta$ and thus we have verified that the classical calculation corresponds to the quantum mechanical one.

A more direct way is to recall the quantum mechanical equations of motion for the expectation values of the operators derived at the beginning of the quarter:

$$\frac{d}{dt} \langle Q \rangle = i \frac{\hbar}{\hbar} \langle [H, Q] \rangle$$  

which is equation 3.71 in the book. Now we have $H = -\omega_L S_z$. Let $Q = S_x$. We have

$$\frac{d}{dt} \langle S_x \rangle = i \frac{\hbar}{\hbar} \langle [H, S_x] \rangle = -i \frac{\hbar}{\hbar} \omega_L \langle [S_z, S_x] \rangle$$

$$= -i \frac{\hbar}{\hbar} \omega_L (\hbar) \langle S_y \rangle$$

$$= \omega_L \langle S_y \rangle$$

which is exactly the classical equation of motion.