Lecture 2
Binomial and Poisson Probability Distributions

Binomial Probability Distribution

Consider a situation where there are only two possible outcomes (a Bernoulli trial)

Example:
- flipping a coin
  - head or tail
- rolling a dice
  - 6 or not 6 (i.e. 1, 2, 3, 4, 5)

Label the possible outcomes by the variable $k$

find the probability $P(k)$ for event $k$ to occur

Since $k$ can take on only 2 values we define those values as:
- $k = 0$ or $k = 1$

let $P(k = 0) = q$ (remember $0 \leq q \leq 1$)

something must happen

$P(k = 0) + P(k = 1) = 1$

$P(k = 1) = p = 1 - q$

write the probability distribution $P(k)$ as:

$P(k) = p^k q^{1-k}$ (Bernoulli distribution)

coin toss: define probability for a head as $P(1)$

$P(1) = 0.5$

dice rolling: define probability for a six to be rolled as $P(1)$

$P(1) = 1/6$

$P(0) = 5/6$ (not a six)
What is the mean ($\mu$) of $P(k)$?
\[
\mu = \sum_{k=0}^{1} k P(k) = \sum_{k=0}^{1} k \cdot P(k) = 0 \cdot q + 1 \cdot p = p
\]

What is the Variance ($\sigma^2$) of $P(k)$?
\[
\sigma^2 = \sum_{k=0}^{1} k^2 P(k) - \mu^2 = \sum_{k=0}^{1} k^2 \cdot P(k) - p^2 = 0^2 \cdot P(0) + 1^2 \cdot P(1) - p^2 = p \cdot p^2 = p(1 - p) = pq
\]

Suppose we have $N$ trials (e.g. we flip a coin $N$ times)
- what is the probability to get $m$ successes (= heads)?

Consider tossing a coin twice. The possible outcomes are:
- no heads: $P(m = 0) = q^2$
- one head: $P(m = 1) = qp + pq$ (toss 1 is a tail, toss 2 is a head or toss 1 is head, toss 2 is a tail) = $2pq$
  - two outcomes because we don't care which of the tosses is a head
- two heads: $P(m = 2) = p^2$
- $P(0) + P(1) + P(2) = q^2 + 2pq + p^2 = (q + p)^2 = 1$

We want the probability distribution function $P(m, N, p)$ where:
- $m$ = number of success (e.g. number of heads in a coin toss)
- $N$ = number of trials (e.g. number of coin tosses)
- $p$ = probability for a success (e.g. 0.5 for a head)
If we look at the three choices for the coin flip example, each term is of the form:

\[ C_m p^m q^{N-m} \quad m = 0, 1, 2, N = 2 \text{ for our example, } q = 1 - p \text{ always!} \]

- The coefficient \( C_m \) takes into account the number of ways an outcome can occur regardless of order.
- For \( m = 0 \) or \( 2 \) there is only one way for the outcome (both tosses give heads or tails): \( C_0 = C_2 = 1 \)
- For \( m = 1 \) (one head, two tosses) there are two ways that this can occur: \( C_1 = 2 \).

- Binomial coefficients: number of ways of taking \( N \) things \( m \) at time

\[ C_{N,m} = \binom{N}{m} = \frac{N!}{m!(N \Box m)!} \]

- \( 0! = 1! = 1, 2! = 1 \cdot 2 = 2, 3! = 1 \cdot 2 \cdot 3 = 6, m! = 1 \cdot 2 \cdot 3 \ldots m \)

- Order of things is not important
  - e.g. 2 tosses, one head case (\( m = 1 \))
    - we don't care if toss 1 produced the head or if toss 2 produced the head

- Unordered groups such as our example are called combinations

- Ordered arrangements are called permutations

For \( N \) distinguishable objects, if we want to group them \( m \) at a time, the number of permutations:

\[ P_{N,m} = \frac{N!}{(N \Box m)!} \]

- example: If we tossed a coin twice (\( N = 2 \)), there are two ways for getting one head (\( m = 1 \))

- example: Suppose we have 3 balls, one white, one red, and one blue.
  - Number of possible pairs we could have, keeping track of order is 6 (rw, wr, rb, br, wb, bw):
    \[ P_{3,2} = \frac{3!}{(3 \Box 2)!} = 6 \]

- If order is not important (rw = wr), then the binomial formula gives
  \[ C_{3,2} = \frac{3!}{2!(3 \Box 2)!} = 3 \quad \text{number of two-color combinations} \]
Binomial distribution: the probability of $m$ success out of $N$ trials:

$$P(m, N, p) = C_{N,m} p^m q^{N-m} = \binom{N}{m} p^m q^{N-m} = \frac{N!}{m!(N-m)!} p^m q^{N-m}$$

- $p$ is probability of a success and $q = 1 - p$ is probability of a failure

Consider a game where the player bats 4 times:

- probability of $0/4 = (0.67)^4 = 20\%$
- probability of $1/4 = \frac{4!}{(3!1!)}(0.33)(0.67)^3 = 40\%$
- probability of $2/4 = \frac{4!}{(2!2!)}(0.33)^2(0.67)^2 = 29\%$
- probability of $3/4 = \frac{4!}{(1!3!)}(0.33)^3(0.67)^1 = 10\%$
- probability of $4/4 = \frac{4!}{(0!4!)}(0.33)^4(0.67)^0 = 1\%$
- probability of getting at least one hit $= 1 - P(0) = 0.8$
To show that the binomial distribution is properly normalized, use Binomial Theorem:

\[(a + b)^k = \sum_{l=0}^{k} \binom{k}{l} a^l b^{k-l}\]

\[
\sum_{m=0}^{N} P(m, N, p) = \sum_{m=0}^{N} \binom{N}{m} p^m q^{N-m} = (p + q)^N = 1
\]

The binomial distribution is properly normalized.

Mean of binomial distribution:

\[
\sum_{m=0}^{N} mP(m, N, p) \over \sum_{m=0}^{N} P(m, N, p) = \sum_{m=0}^{N} m \binom{N}{m} p^m q^{N-m}
\]

A cute way of evaluating the above sum is to take the derivative:

\[
\frac{\partial}{\partial p} \sum_{m=0}^{N} \binom{N}{m} p^m q^{N-m} = 0
\]

\[
\sum_{m=0}^{N} m \binom{N}{m} p^m q^{N-m} = \sum_{m=0}^{N} \binom{N}{m} p^m (N-m)(1-p)^{N-m} = 0
\]

\[
p \sum_{m=0}^{N} \binom{N}{m} p^m q^{N-m} = N \sum_{m=0}^{N} \binom{N}{m} p^m (1-p)^{N-m} = N \sum_{m=0}^{N} \binom{N}{m} p^m (1-p)^{N-m}
\]

\[
p \sum_{m=0}^{N} \binom{N}{m} p^m q^{N-m} = N \sum_{m=0}^{N} \binom{N}{m} p^m (1-p)^{N-m} = Np
\]

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L2: Binomial and Poisson
Variance of binomial distribution (obtained using similar trick):

\[
\sum_{m=0}^{N} \left( \begin{array}{c} m \\ N \\ \end{array} \right)^2 P(m, N, p) \]

\[
= \frac{N!}{m!(N-m)!} P(m, N, p) \]

\[
= Npq \]

Example: Suppose you observed \( m \) special events (success) in a sample of \( N \) events

- measured probability ("efficiency") for a special event to occur:
  \[
  = \frac{m}{N} \]

- error on the probability ("error on the efficiency"):
  \[
  \sigma = \frac{ \sqrt{Npq} }{ \sqrt{Np(1-p)} } = \sqrt{\frac{Npq}{Np(1-p)}} = \sqrt{ \frac{1}{N} } \]

- sample \((N)\) should be as large as possible to reduce uncertainty in the probability measurement

Example: Suppose a baseball player's batting average is 0.33 (1 for 3 on average).

- Consider the case where the player either gets a hit or makes an out (forget about walks here!).
  - probability for a hit: \( p = 0.33 \)
  - probability for "no hit": \( q = 1 - p = 0.67 \)

- On average how many hits does the player get in 100 at bats?
  \[
  \bar{x} = Np = 100 \cdot 0.33 = 33 \text{ hits} \]

- What's the standard deviation for the number of hits in 100 at bats?
  \[
  \sigma = (Npq)^{1/2} = (100 \cdot 0.33 \cdot 0.67)^{1/2} \approx 4.7 \text{ hits} \]

- we expect \( \approx 33 \pm 5 \) hits per 100 at bats
Poisson Probability Distribution

- A widely used discrete probability distribution
- Consider the following conditions:
  - $p$ is very small and approaches 0
    - example: a 100 sided dice instead of a 6 sided dice, $p = 1/100$ instead of $1/6$
    - example: a 1000 sided dice, $p = 1/1000$
  - $N$ is very large and approaches $\infty$
    - example: throwing 100 or 1000 dice instead of 2 dice
  - product $Np$ is finite
- Example: radioactive decay
  - Suppose we have 25 mg of an element
    - very large number of atoms: $N \approx 10^{20}$
  - Suppose the lifetime of this element $\tau = 10^{12}$ years $\approx 5 \times 10^{19}$ seconds
    - probability of a given nucleus to decay in one second is very small: $p = 1/\tau = 2 \times 10^{-20}$/sec
    - $Np = 2$/sec finite!
  - number of counts in a time interval is a Poisson process
- Poisson distribution can be derived by taking the appropriate limits of the binomial distribution
  \[
P(m, N, p) = \frac{N!}{m!(N - m)!} p^m q^{N-m}
  \]
  \[
  \frac{N!}{(N - m)!} = \frac{N(N - 1) \cdots (N - m + 1)(N - m)!}{(N - m)!} = N^m
  \]
  \[
  q^{N-m} = (1 - p)^N = 1 - p(N - m) + \frac{p^2(N - m)(N - m - 1)}{2!} + \cdots + \frac{(pN)^2}{2!} + \cdots e^{-pN}
  \]

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L2: Binomial and Poisson
\[ P(m, N, p) = \frac{N^m}{m!} p^m e^{-pN} \]

Let \( \Box = Np \)

\[ P(m, \Box) = \frac{\Box^m}{m!} e^{-\Box} \]

\[ \sum_{m=0}^\infty \frac{\Box^m}{m!} = e^\Box = e^{Np} = 1 \]

- \( m \) is always an integer \( \geq 0 \)
- \( \Box \) does not have to be an integer
- It is easy to show that:
  - \( \Box = Np = \) mean of a Poisson distribution
  - \( \Box^2 = Np = \) variance of a Poisson distribution
  - Radioactivity example with an average of 2 decays/sec:
    - What’s the probability of zero decays in one second?
      \[ p(0, 2) = \frac{e^{2} \cdot 0^0}{0!} = \frac{e^{2} \cdot 1}{1} = e^{2} = 0.135 \] 13.5%
    - What’s the probability of more than one decay in one second?
      \[ p(> 1, 2) = 1 - p(0, 2) - p(1, 2) = 1 - \frac{e^{2} \cdot 0^0}{0!} - \frac{e^{2} \cdot 2^1}{1!} = 1 - e^{2} \cdot 2e^{2} = 0.594 \] 59.4%
    - Estimate the most probable number of decays/sec?
      \[ \frac{\partial}{\partial m} P(m, \Box) \bigg|_{m^*} = 0 \]

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L2: Binomial and Poisson
To solve this problem its convenient to maximize \( \ln P(m, \Box) \) instead of \( P(m, \Box) \).

\[
\ln P(m, \Box) = \ln \frac{e^{m \ln m} \Box}{m!} = \Box + m \ln \Box \ln m!
\]

In order to handle the factorial when take the derivative we use Stirling's Approximation:

\[
\ln m! \approx m \ln m - m 
\]

\[
\frac{\partial}{\partial m} \ln P(m, \Box) = \frac{\partial}{\partial m} (\Box + m \ln \Box \ln m!)
\]

\[
= \ln \Box \ln m \Box m \frac{1}{m} + 1 
\]

\[
= 0
\]

\[
m^* = \Box
\]

The most probable value for \( m \) is just the average of the distribution

If you observed \( m \) events in an experiment, the error on \( m \) is

\[
\Box = \sqrt{\Box} = \sqrt{m}
\]

This is only approximate since Stirlings Approximation is only valid for large \( m \).

Strictly speaking \( m \) can only take on integer values while \( \Box \) is not restricted to be an integer.
Comparison of Binomial and Poisson distributions with mean $\mu = 1$

For large $N$: Binomial distribution looks like a Poisson of the same mean.