4-vectors, notations

Defining \( x^0 = ct \), \( x^1 = x \), \( x^2 = y \), \( x^3 = z \), write Lorentz transformation as

\[
\begin{pmatrix}
  x'^0 \\
  x'^1 \\
  x'^2 \\
  x'^3
\end{pmatrix} = \begin{pmatrix}
  \gamma & -\beta \gamma & 0 & 0 \\
  -\beta \gamma & \gamma & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
  x^0 \\
  x^1 \\
  x^2 \\
  x^3
\end{pmatrix}
\]

\( \gamma = \frac{1}{\sqrt{1 - \beta^2}} \) and \( \beta = \frac{v}{c} \).

**Definition** A 4-vector \( A^\mu = \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} \) is an object which under Lorentz transformation transforms as

\[
\begin{pmatrix}
  A'^0 \\
  A'^1 \\
  A'^2 \\
  A'^3
\end{pmatrix} = \begin{pmatrix}
  \gamma & -\beta \gamma & 0 & 0 \\
  -\beta \gamma & \gamma & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
  A^0 \\
  A^1 \\
  A^2 \\
  A^3
\end{pmatrix}
\]

(e.g., \( x^\mu \) is a contravariant 4-vector.)

\( A^\mu \) is a **contravariant vector**:

\[
A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu
\]

\( B_\mu \) is a **covariant vector**:

\[
B_\mu' = \frac{\partial x^\nu}{\partial x'^\mu} B_\nu
\]

\( \frac{\partial y}{\partial x^\mu} \equiv \nabla_\mu y \) with \( y \), scalar field is a covariant vector.
as \[ \frac{\partial \phi}{\partial x^\mu} = \frac{\partial x^\nu}{\partial x^\mu} \frac{\partial \phi}{\partial x^\nu} \].

**Tensors:** \( A^\mu B_\nu \) - contravariant, \( A_\mu B^\nu \) - covariant (rank 2), can have higher ranks.

**Def.** Scalar (inner) product of 2 vectors is \( A_\mu B^\mu \). (Assume summation).

It is Lorentz-invariant:
\[ A^\mu B^\nu = \frac{\partial x^\xi}{\partial x'^\mu} A_\xi \frac{\partial x'^\nu}{\partial x^\beta} B^\beta = \frac{\partial x^\alpha}{\partial x^\beta} A_\alpha B^\beta = \delta_{\beta}^\alpha A_\alpha B^\alpha. \]

**Def.** The interval \( ds^2 = dx_\mu dx_\mu = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \).

It is a Lorentz-invariant too.

**Def.** The metric tensor \( g_{\mu\nu} \) is defined by
\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu \]

In our Minkowski space \( g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \equiv \eta_{\mu\nu} \)

(Throughout the course we'll use this notation.)

\( dx_\mu dx_\nu \) is also a Lorentz-scalar \( \Rightarrow dx_\mu = g_{\mu\nu} dx^\nu \)

\( \Rightarrow g_{\mu\nu} \) lowers & raises indices!
Example: \( x^\mu = (ct, \vec{x}) \Rightarrow x_\mu = g_{\mu\nu} x^\nu = (ct, -\vec{x}) \)

Contravariant

In general
\[ A_\mu = g_{\mu\nu} A^\nu, \quad A^\mu = g^{\mu\nu} A_\nu \]

where \( g^{\mu\nu} \) is defined by requiring that
\[ g^{\mu\alpha} g_{\alpha\beta} = \delta^{\mu}_{\beta} : \text{ if that is true} \Rightarrow \text{start with} \]
\[ A_\mu = g_{\mu\nu} A^\nu \Rightarrow g^{\alpha\mu} A_\alpha = g^{\alpha\mu} g_{\mu\nu} A^\nu = \delta^{\alpha}_{\nu} A^\nu = A^\alpha \Rightarrow A^\mu = g^{\mu\alpha} A_\alpha. \]

\( g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \eta^{\mu\nu} + \epsilon^{\mu\nu}. \quad (= g_{\mu\nu}) \)

Def. \( \partial_\mu \equiv \frac{2}{c^2} \frac{d}{dt} \), \( \partial^\mu \equiv \frac{2}{c^2} \frac{d}{dx^\mu} \Rightarrow \partial_\mu \phi \) is covariant vector,
\( \partial^\mu \phi \) is a contravariant vector. (check!)

\( \partial_\mu A^\mu \) is a Lorentz-invariant.

\( \partial_\mu \partial^\mu = \frac{1}{c^2} \frac{2^2}{c^2} - \nabla^2 \) is also Lorentz-invariant.

Examples: other important 4-vectors are
\[ p^\mu = \begin{pmatrix} \frac{\epsilon}{c} \\ \vec{p} \end{pmatrix}, \quad \rho^\mu = \begin{pmatrix} \frac{\epsilon}{c} \\ -\vec{\rho} \end{pmatrix} \Rightarrow \rho_\mu \rho^\mu = \left( \frac{\epsilon}{c} \right)^2 - \vec{\rho}^2 = m^2 c^2. \]
$A^\mu = (\Phi, \vec{A})$ in $E \cdot m$, $\Phi$ - electric potential, $\vec{A}$ - vector potential.

$J^\mu = (j, \vec{j})$ with $j$ the charge density, $\vec{j}$ the current density.

Notations from now on $c = 1$ and $\hbar = 1$ "natural units":

$\Rightarrow$ mass, momentum, energy are measured in the same units (eV, keV, MeV, GeV, ...)

$1\text{eV} = 1.6 \times 10^{-19} J$

distances, time are measured in femto-meters aka femtis (f m):

$1\text{fm} = 5 \text{ GeV}^{-1}$

$1\text{GeV} = 10^9 \text{eV}$, $1$ femtometer $= 10^{-15}$ m. $= 1$ fm.

proton's mass $m_p = 0.938 \text{ GeV} \approx 1 \text{GeV}$

electron's mass $m_e = 0.511 \text{ MeV} = 0.5 \times 10^{-3} \text{ GeV}$
Classical Scalar Field Theory (real field)

\[ \Phi(x^\mu) = \Phi(x^0, \vec{x}) \] 

a function of space-time points \( x^\mu \)

Example: temperature field \( T(t, \vec{x}) \)

In classical mechanics one has point particles \( i = 1, \ldots, N \) with the Lagrangian \( L(\dot{q}^i, \ddot{q}^i, t) \)

and the action \( S = \int dt \ L(\dot{q}^i, \ddot{q}^i, t) \)

\( q^i \) degrees of freedom (e.g. particle coordinates)

\( \dot{q}^i = \frac{dq^i}{dt} \) generalized velocities

Now, instead of discrete point particles we have a field \( \Phi(x, t) \rightarrow \)

Classical Mechanics  |  Classical Field Theory
--- | ---
\( q^i \)  |  \( \Phi(x^0, \vec{x}) \)
\( i \)  |  \( \vec{x}, t \)
\( \dot{q}^i \)  |  \( \partial_\mu \Phi, \mu = 0, 1, 2, 3 \)
\( L(\dot{q}^i, \ddot{q}^i, t) \)  |  \( \int d^3x \ L(\Phi, \partial_\mu \Phi) \)
$L$ is Lagrangian density. (usually called the lagrangian)

The action is $S = \int dt \ L = \int dt \ d^3x \ \frac{L (\varphi, \partial_\mu \varphi)}{d^3x}$ (remember $c=1$)

$S$ is a Lorentz scalar (better be, physics is Lorentz invariant)

What about $d^4x = dx^0 dx' dx^2 dx^3$? Remember that

$x'^\mu = \Lambda^\mu_\nu x^\nu$ with $\Lambda^\mu_\nu = \frac{\partial x'^\mu}{\partial x^\nu}$ a matrix of $\mathbb{L}^{4,2}$.

$\Rightarrow d^4x' = \det \Lambda \cdot d^4x$.

Jacobi

Now, $\det \Lambda = \det \begin{pmatrix} \delta & -\beta x & 0 & 0 \\ -\beta \gamma & \delta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \delta^2 (1 - \beta^2) = 1$ (true in general)

$\Rightarrow d^4x' = d^4x \Rightarrow d^4x$ is a Lorentz scalar

$\Rightarrow L$ is a Lorentz scalar!

Just like in classical mechanics, in classical field theory dynamics is given by the least action principle: field $\varphi$ is determined by requiring that $S$ is stationary with respect to small perturbations around $\varphi$: $S [\varphi + \delta \varphi] = S [\varphi] + o (\delta \varphi^2)$.
\[ 0 = \mathcal{L} = \int d^4x \left[ \frac{\delta \mathcal{L}}{\delta \varphi} \varphi + \frac{\delta \mathcal{L}}{\delta (\partial_\mu \varphi)} \partial_\mu \varphi \right] = \]
\[ = (\text{as } \delta \varphi = \partial_\mu \varphi \Rightarrow \text{parts}) = \int d^4x \left[ \frac{\delta \mathcal{L}}{\delta \varphi} \varphi - \partial_\mu \left( \frac{\delta \mathcal{L}}{\delta (\partial_\mu \varphi)} \right) \right] + \text{surface term} \]
\[ = \int d^4x \varphi \left[ \frac{\delta \mathcal{L}}{\delta \varphi} - \partial_\mu \left( \frac{\delta \mathcal{L}}{\delta (\partial_\mu \varphi)} \right) \right] + \text{surface term} \]
\[ \text{Euler-Lagrange equations (aka equations of motion) for field } \varphi. \]  
\[
\Rightarrow \left[ \frac{\delta \mathcal{L}}{\delta \varphi} - \partial_\mu \left( \frac{\delta \mathcal{L}}{\delta (\partial_\mu \varphi)} \right) = 0 \right]
\]

Now, \( \varphi(x) \) is a scalar field \( \Rightarrow \) it is Lorentz-inv., which means that: \( \varphi(x) \rightarrow \varphi'(x') = \varphi(x) \)

\[ \Rightarrow \text{as } x'^\mu = \Lambda^\mu_\nu x^\nu \Rightarrow \Lambda^\nu_\mu x \Rightarrow \varphi'(x) = \varphi(\Lambda^\nu_\mu x). \]

Lagrangian density for massive scalar field:

\[ \mathcal{L} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{m^2}{2} \varphi^2. \]

EOM: \( \frac{\delta \mathcal{L}}{\delta \varphi} = -m^2 \varphi; \quad \frac{\delta \mathcal{L}}{\delta (\partial_\mu \varphi)} = \partial_\mu \varphi \Rightarrow \)

\[ \Rightarrow \partial_\mu \left( \frac{\delta \mathcal{L}}{\delta (\partial_\mu \varphi)} \right) = \partial_\mu \partial^\mu \varphi \Rightarrow -m^2 \varphi - \partial_\mu \partial^\mu \varphi = 0 \]
\[ [\partial^2 + m^2] \psi = 0 \]

or

\[ [\Box + m^2] \psi = 0 \]

To solve K-G equation write \( \psi(x) = \int d^4 k \; e^{-i k \cdot x} \tilde{\psi}(k) \)

with \( k \cdot x = k^\mu x^\mu = k^0 x^0 - \vec{k} \cdot \vec{x} \).

\[ [\Box + m^2] \psi = \int d^4 k \; \tilde{\psi}(k) (\Box + m^2) e^{-i k \cdot x} = \int d^4 k \; \tilde{\psi}(k) \]

\[ [ - k^2 + m^2 ] = 0 \quad \text{with} \quad k^2 = k^\mu k_\mu = (k^0)^2 - (\vec{k})^2 \]

\[ \Rightarrow [ k^2 - m^2 ] \tilde{\psi} = 0 \Rightarrow as \; \tilde{\psi} \neq 0 \Rightarrow k^2 = m^2 \quad or \]

\[ E_k^2 - k^2 = m^2 \Rightarrow E_k = \pm \sqrt{k^2 + m^2} \quad \Rightarrow \text{define} \quad E_k = \sqrt{k^2 + m^2} \]

\[ \Rightarrow \psi(x) = \int \frac{d^3 k}{(2\pi)^3 2 E_k} \left[ a_k^+ e^{-i E_k t + i \vec{k} \cdot \vec{x}} + a_k e^{i E_k t - i \vec{k} \cdot \vec{x}} \right] \]

most general solution