Classification of Physical Quantities by Space-Time Symmetries

A. Spatial Rotations

\[ X_\alpha \rightarrow \underbrace{X'_\alpha = R_{\alpha \beta} X_\beta} \]

under rotation \( \Rightarrow (R_{\alpha \beta} X_\beta) (R_{\nu \delta} X_\delta) = X_\alpha X_\alpha \)

\[ \Rightarrow (R_{\alpha \beta} R_{\nu \delta} = \delta_{\nu \beta} \Rightarrow (R^{-1})_{\nu \delta} = R_{\nu \beta} \]

\[ \det R = +1 \sim \text{rotation w/o reflection} \]

vectors: \( A_\alpha \rightarrow A'^\alpha = R_{\alpha \beta} X_\beta \)

Tensors: \( T_{\alpha_1 \alpha_2 \ldots \alpha_n} \rightarrow T'^1_{\alpha_1 \alpha_2 \ldots \alpha_n} = R_{\alpha_1 \beta_1} R_{\alpha_2 \beta_2} \ldots R_{\alpha_n \beta_n} \) (definition)

B. Spatial Reflection

\[ \vec{x} \rightarrow -\vec{x} \sim \text{all vectors (or polar vectors)} \]

transforms like this

\[ \vec{z} = \vec{x} \times \vec{y} \Rightarrow \vec{x} \rightarrow -\vec{x}, \vec{y} \rightarrow -\vec{y} \]

\[ \Rightarrow \vec{z} \rightarrow -\vec{z} \]

axial vector (pseudovector)
Inversion is also called parity \( P \).

\[ P \text{ vector } \rightarrow - \text{ vector} \quad \text{axial vector } \rightarrow \text{axial vector} \]

\[ P = -1 \quad P = +1 \]

Tensor of rank \( N \): \( P \alpha_1 \cdots \alpha_N = (-1)^N T_{\alpha_1 \cdots \alpha_N} \)

Pseudotensor of rank \( N \): \( P \alpha_1 \cdots \alpha_N = (-1)^{N+1} T_{\alpha_1 \cdots \alpha_N} \)

\[ \text{E.g.} \quad \vec{Z} = \vec{x} \times \vec{y} \Rightarrow Z_\alpha = \varepsilon_{\alpha \beta \gamma} x_\beta y_\gamma \Rightarrow \varepsilon_{\alpha \beta \gamma} \text{ has } P = +1 \]

\[ \begin{array}{c|c}
P = 1 & P = -1 \\
\hline
p = 1 & p = -1
\end{array} \]

\[ \Rightarrow p = (-1)^{N+1} \Rightarrow \varepsilon_{\alpha \beta \gamma} \text{ is pseudotensor.} \]

C. Time reversal: \( t \rightarrow -t \)

\[ \Pi \vec{x} = -\vec{x} \quad , \quad \Pi \vec{p} = -\vec{p} \rightarrow \Pi \vec{\rho} = -\vec{\rho} \text{ is } T\text{-odd.} \]

\[ \uparrow \text{ T-even.} \]

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Tensor Rank</th>
<th>Parity</th>
<th>Time Reversal</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \vec{x} )</td>
<td>vector</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>( \vec{v} = \frac{dx}{dt} )</td>
<td>vector</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>( \vec{p} )</td>
<td>vector</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>( \vec{L} = \vec{x} \times \vec{p} )</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>( \vec{F} = m\vec{a} )</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>( \vec{N} = \vec{x} \times \vec{F} )</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Energy</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
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<td>Quantity</td>
<td>Tensor Rank</td>
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</tr>
<tr>
<td>-------------------</td>
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<tr>
<td>$\mathbf{F}$</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\mathbf{F}^2$ (= $\mathbf{F} \cdot \mathbf{F}$)</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$\mathbf{E}$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$\mathbf{B}$</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\mathbf{D}$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\mathbf{B}$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
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<tr>
<td>$\mathbf{D}$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\mathbf{H}$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\mathbf{S} = \mathbf{E} \times \mathbf{H}$</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$T^\alpha_\beta$</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Consider a half-infinite ideal solenoid; it has current I and N loops per unit length. Each loop carries magnetic moment

\[ \hat{m} = I \pi a^2 \hat{z} \]

Assume that a distant observer sees

\[ \vec{A}(x) \approx \frac{\mu_0}{4\pi} \int \frac{dm'}{1 \cdot \vec{x} - \vec{x}' \cdot 1^3} \]

where \( dm = I \pi a^2 \ N \ dx \)
\[ \vec{A}(\vec{x}) = \frac{\Lambda_0}{4\pi} \alpha^2 I N \int_{-\infty}^{\infty} d^2 \tau \frac{\hat{\tau} \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} = \]

\[ = -\frac{\Lambda_0}{4\pi} \alpha^2 I N \int_{-\infty}^{\infty} d^2 \tau \hat{\tau} \times \nabla \frac{1}{|\vec{x} - \vec{x}'|} \]

\[ \vec{B}(\vec{x}) = \nabla \times \vec{A} = -\frac{\Lambda_0}{4\pi} \alpha^2 I N \int_{-\infty}^{\infty} d^2 \tau \nabla \times \left( \hat{\tau} \times \frac{1}{|\vec{x} - \vec{x}'|} \right) = \]

\[ = -\frac{\Lambda_0}{4\pi} \alpha^2 I N \int_{-\infty}^{\infty} d^2 \tau \left[ \hat{\tau} \cdot \nabla^2 \frac{1}{|\vec{x} - \vec{x}'|} - \nabla \cdot \frac{1}{|\vec{x} - \vec{x}'|} \right] = \]

\[ = -\frac{\Lambda_0}{4\pi} \alpha^2 I N \hat{\tau} \delta(\vec{x} - \vec{x}') \]

\[ = \frac{\Lambda_0}{4\pi} \alpha^2 I N \hat{\tau} \delta(x) \delta(y) \Theta(-z) - \]

\[ -\frac{\Lambda_0}{4\pi} \alpha^2 I N \nabla \cdot \frac{1}{|\vec{x}|} \]

\[ \Rightarrow \vec{B} = \frac{\Lambda_0}{4\pi} \alpha^2 I N \hat{\tau} \delta(x) \delta(y) \Theta(-z) + \]

\[ + \frac{\Lambda_0}{4\pi} \alpha^2 I N \frac{x}{|x|^3} \]

Dirac "monopole": take \( a \to 0 \)

\[ \vec{B} = \frac{g}{4\pi} \frac{x}{|x|^3} \Rightarrow \text{Independence of the solenoid's shape imposed by quantum mechanics} \]
Deform the solenoid: \( g = \frac{e A_0}{2} \) \( \Rightarrow \quad \vec{A}'(\vec{x}) = g \oint d\vec{x} \times \frac{\vec{x} - \vec{x}'}{1 - \vec{x} \cdot \vec{x}'} \) 

\[
\vec{A}' - \vec{A} = g \oint_{\gamma} d\vec{x} \times \frac{\vec{x} - \vec{x}'}{1 - \vec{x} \cdot \vec{x}'} = -g \oint_{\gamma} d\vec{x} \times \frac{\vec{x}}{1 - \vec{x} \cdot \vec{x}'} = \]

\[
= g \oint_{\gamma} d\vec{a} \times \vec{\nabla} \frac{\vec{x} - \vec{x}'}{1 - \vec{x} \cdot \vec{x}'} = -g \oint_{\gamma} d\vec{a} \times \vec{\nabla} \frac{\vec{x} - \vec{x}'}{1 - \vec{x} \cdot \vec{x}'} =
\]

\[
= g \oint_{\gamma} d\vec{a'} \cdot \vec{\nabla} \frac{\vec{x} - \vec{x}'}{1 - \vec{x} \cdot \vec{x}'} =
\]

\[
d\Omega = \frac{d\Omega'}{1 - \vec{x} \cdot \vec{x}'} = \Rightarrow \Omega \text{ is the solid angle subtended by } \gamma'
\]

\[
\Rightarrow \quad \vec{A}' = \vec{A} + \frac{g}{\gamma} \vec{\nabla} \phi \quad \text{is the gauge transform of } \vec{A} \text{ in physics}
\]

\[
\text{Fermions: } \psi \rightarrow \psi' = \psi e^{i \frac{g \phi}{\hbar}} \quad \text{under gauge transform } \vec{A} \rightarrow \vec{A} + \vec{\nabla} \phi \Rightarrow \psi' = \psi e^{i \frac{g \phi}{\hbar}}
\]

\[
\Rightarrow \text{ if } \phi \rightarrow \phi + \frac{2\pi n}{\hbar} \Rightarrow \text{ for } \phi \text{ to be single-valued we need } \frac{2\pi n}{\hbar} = 2\pi \text{ integer } n \text{ discrete magnetic charge } g
\]