Example: Ferromagnetic sphere:

\[ \mathbf{M} \text{ a constant, } \mathbf{J} = 0, \mathbf{M} = M \mathbf{\hat{z}} \]

Find \( \mathbf{B} \), \( \mathbf{H} \) everywhere.

\[ \nabla \cdot \mathbf{M} = 0 \quad \Rightarrow \quad \Phi = \frac{1}{\gamma a} \oint d\mathbf{a} \cdot \frac{\mathbf{M}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} = \]

\[ = \frac{M}{\gamma a} a^2 \int d\mathbf{x}' \frac{\cos \theta'}{|\mathbf{x} - \mathbf{x}'|} \]

Using \[ \frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{\gamma a} \sum_{l,m} \frac{1}{2l + 1} \frac{V_{l}^{2}}{r_{l+1}^{2l+1}} \ Y_{l,m}^{*}(\theta', \phi') Y_{l,m}(\theta, \phi) \]

and \( Y_{10}(\theta, \phi) = \sqrt{\frac{3}{\gamma a}} \cos \theta \) we get

\[ \Phi = \frac{Ma^2}{\gamma a} \int_0^{2\pi} d\phi' \int_0^{\pi} d\cos \theta' \frac{4\pi}{\gamma a} \sum_{l,m} \frac{1}{2l+1} \frac{V_{l}^{2}}{r_{l+1}^{2l+1}} \]

\[ Y_{l,m}^{*}(\theta', \phi') \cdot \sqrt{\frac{\gamma a}{3}} Y_{10}(\theta', \phi') Y_{l,m}(\theta, \phi) = (\text{only } l=1) \]

\[ = \frac{Ma^2}{3} \frac{V_{l}^{2}}{r_{l}^{2}} \cos \theta' \]

where \( V_{l} = \min \{ r, a \} \)

\[ \Rightarrow \text{inside} \]

\[ \Phi = \frac{1}{3} M r \cos \theta = \frac{1}{3} M z \]

\[ \text{outside} \]

\[ \Phi = \frac{1}{3} \frac{Ma^3}{r^2} \cos \theta \]
\[ \vec{H} = -\nabla \Phi_m \Rightarrow \vec{H} = -\frac{1}{3} \vec{M} \text{ inside} \]
\[ \vec{B} = \mu_0 (\vec{H} + \vec{H}^0) = \frac{2}{3} \vec{M} \text{ inside}. \]

Outside:
\[ \vec{H} = -\nabla \left( \frac{1}{3} \frac{Ma^3}{r^2} \cos \theta \right), \quad \vec{B} = \mu_0 \vec{H}. \]

\[ \Rightarrow \vec{B} = -\frac{1}{3} \mu_0 Ma^3 \nabla \left( \frac{\vec{r} \cdot \vec{r}}{r^3} \right) \Rightarrow \text{dipole moment} \]
\[ \vec{m} = \frac{4\pi a^3}{3} \vec{M} \]

Example: a sphere with permeable magnetic material (\( \vec{B} = \mu_0 \vec{H} \) inside, \( \vec{B} = \mu_0 \vec{H}^0 \) outside) in external magnetic field \( \vec{H}_0 \).

\[ \nabla^2 \Phi_m = 0 \text{ inside \\& outside} \]

\[ \Rightarrow \Phi_{\text{inside}} = \frac{\sum Ae_r e^l P_l (\cos \theta)}{r}, \quad r < a \]

\[ \Phi_{\text{outside}} = -H_0 r P_1 (\cos \theta) + \sum_{l=1}^{\infty} \frac{Be_l}{r^{l+1}} P_l (\cos \theta), \quad r > a \]

As \( H_{in, r} = H_{out, r} \Rightarrow \)

\[ \Rightarrow \frac{\partial}{\partial r} \phi \bigg|_{r=a} = \frac{2}{3} \mu_0 \vec{A}_1 \quad , \quad r = a \]

\[ \frac{\partial}{\partial r} \vec{H}_{in} \bigg|_{r=a} = \mu_0 \frac{\partial}{\partial r} \vec{H}_{out} \bigg|_{r=a} \]

\[ \Rightarrow \begin{align*}
(1) & \quad A_1 \cdot r = -H_0 a + \frac{B_1}{a^2} \\
(2) & \quad \mu_0 A_1 = -\mu_0 H_0 - \frac{2B_1}{a^3} \end{align*} \]
\[- H_0 + \frac{B_1}{a^3} = - \frac{\mu_0}{\mu} H_0 - \frac{2B_1}{a^3} \frac{\mu_0}{\mu} \]

\[\Rightarrow \frac{B_1}{a^3} \left( 1 + 2 \frac{\mu_0}{\mu} \right) = H_0 \left( 1 - \frac{\mu_0}{\mu} \right) \Rightarrow B_1 = H_0 \frac{\mu - \mu_0}{\mu + 2\mu_0} \frac{a^3}{\mu + 2\mu_0} \]

\[A_1 = -H_0 + \frac{B_1}{a^3} \Rightarrow A_1 = -\frac{3\mu_0}{\mu + 2\mu_0} H_0 \]

\[\Rightarrow \Phi_{\text{inside}} = -\frac{3\mu_0}{\mu + 2\mu_0} H_0 r \cos \Theta = -\frac{3\mu_0}{\mu + 2\mu_0} H_0 z \]

\[\Phi_{\text{outside}} = -H_0 \cos \Theta + H_0 a^3 \frac{M - \mu_0}{\mu + 2\mu_0} \frac{1}{r^2} \cos \Theta \]

& one can find

\[\vec{H}_{\text{inside}} = -\nabla \Phi_{\text{inside}} = \frac{3\mu_0}{\mu + 2\mu_0} \frac{\vec{H}_0}{\mu + 2\mu_0} \]

\[\vec{B}_{\text{inside}} = \mu \vec{H}_{\text{inside}} \]

Effective magnetization: \[\vec{M} = \frac{1}{\mu_0} \vec{B}_{\text{inside}} - \vec{H}_{\text{inside}} = \left( \frac{\mu}{\mu_0} - 1 \right) \vec{H}_{\text{inside}} \Rightarrow \vec{M} = \frac{3}{\mu_0} \frac{M - \mu_0}{\mu + 2\mu_0} \vec{H}_0 \]

\[(5.115)\]

Example: Cylinderical bar magnet:

\[\vec{M} \]

\[\text{find } \vec{B}, \vec{H} \]

\[\Phi_m = \frac{1}{4\pi} \oint_S d\vec{a} \cdot \frac{\hat{n} \cdot \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|} = 0 \]

\[\Rightarrow \vec{H} = 0, \quad \vec{B}_{\text{inside}} = \mu_0 (\vec{H} + \vec{M}) = \mu_0 \vec{M}, \quad \vec{B}_{\text{outside}} = 0.\]
Faraday's Law of Induction.

Suppose we have a current loop \( C \) in an external magnetic field. Faraday observed that changes in \( \vec{B} \) generate current in the loop.

Namely he observed that, if we define magnetic flux

\[
\Phi = \int \vec{B} \cdot d\vec{A},
\]

Then

\[
\mathbf{E} = \oint \vec{E}' \cdot d\vec{e} = -k \frac{d\Phi}{dt}
\]

where \( \mathbf{E} \) is called electromotive force.

\( k \) is a proportionality constant, \( k = 1 \) in SI units. The "-" sign is due to Lenz's law - the system tries to oppose changes.

\( \vec{E}' \) - electric field at the element \( d\vec{e} \) in its rest frame (1)

\( k \) is fixed from Galilean invariance:

\( \Phi \) changes either due to

(i) changes in \( \vec{B} \)

or

(ii) changes in current \( \dot{I} \) or motion of \( C \).
\[
\frac{d\vec{B}}{dt} = \frac{\partial \vec{B}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{B} = \frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times (\vec{B} \times \vec{v}) + \vec{v} (\vec{\nabla} \cdot \vec{B})
\]

\[
= \frac{d}{dt} \int_S \vec{B} \cdot \hat{n} \, d\alpha = \int_S \frac{\partial \vec{B}}{\partial t} \cdot \hat{n} \, d\alpha + \int_S d\alpha \cdot \vec{\nabla} \times (\vec{B} \times \vec{v}) = \int_S \frac{\partial \vec{B}}{\partial t} \cdot \hat{n} \, d\alpha + \oint_C d\vec{l} \times (\vec{B} \times \vec{v})
\]

\[
\Rightarrow \text{Faraday's law gives}
\]

\[
\mathcal{E} = \oint_C \vec{E}' \cdot d\vec{l} = -k \int_S \frac{\partial \vec{B}}{\partial t} \cdot \hat{n} \, d\alpha - k \oint_C d\vec{l} \times (\vec{B} \times \vec{v})
\]

\[
\Rightarrow \oint_C d\vec{l} \times (\vec{E}' - k (\vec{v} \times \vec{B})) = -k \int_S \frac{\partial \vec{B}}{\partial t} \cdot \hat{n} \, d\alpha
\]

This is Faraday's law for a moving circuit.

If we consider the case such that the circuit is at rest, in this situation we have

\[
\oint_C \vec{E} \cdot d\vec{l} = -k \int_S \frac{\partial \vec{B}}{\partial t} \cdot \hat{n} \, d\alpha
\]

\[
\Rightarrow \boxed{\vec{E}' = \vec{E} + k \vec{v} \times \vec{B}}
\]

as \( F = q \vec{E} \) and \( F = q \vec{v} \times \vec{B} \) \( \Rightarrow k = 1 \).
\[ \oint \vec{E} \cdot d\vec{S} = \iint (\nabla \times \vec{E}) \cdot \hat{n} \, da = -\int \frac{\partial \vec{B}}{\partial t} \cdot \hat{n} \, da \]

\[ \Rightarrow \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{(differential form of Faraday's law (generalized \( \vec{\nabla} \times \vec{E} = 0 \) in electrostatics))} \]

**Energy in the Magnetic Field**

Energy change rate is (for a point charge)

\[ \frac{dW}{dt} = \vec{v} \cdot \vec{F} = q \vec{v} \cdot \vec{E} \]

(point charge \( q \) moving with velocity \( \vec{v} \))

\[ \Rightarrow \quad SW = -St \cdot \vec{J} \cdot \vec{E} \]

Let's change \( \vec{B} \) by \( S\vec{B}(x) \):

\[ SW = -St \int d^3x \cdot \vec{J} \cdot \vec{E} = -St \int d^3x \left( \nabla \times \vec{H} \right) \cdot \vec{E} \]

As \( \nabla (\vec{E} \times \vec{H}) = \vec{H} \cdot (\nabla \times \vec{E}) - \vec{E} \cdot \nabla \times \vec{H} \)

\[ SW = St \int d^3x \left\{ \nabla \left( \vec{E} \times \vec{H} \right) - \vec{H} \cdot \nabla \times \vec{E} \right\} \]

Surface integral

\[ = -St \int d^3x \vec{H} \cdot (\nabla \times \vec{E}) \Rightarrow \quad \omega S \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]
\[ SW = \int d^3x \, \vec{H} \cdot S \vec{B} \]

If \( \vec{B} = \mu \vec{H} \) \( \implies \)
\[ W = \frac{1}{2} \int d^3x \, \vec{H} \cdot \vec{B} \]

Alternatively, as \( \vec{B} = \nabla \times \vec{A} \) \( \implies \)
\[ W = \frac{1}{2} \int d^3x \, \vec{H} \cdot (\nabla \times \vec{A}) = \frac{1}{2} \int d^3x \, \vec{A} \cdot (\nabla \times \vec{H}) \]
\[ \frac{\vec{J}}{J} \]

\[ \implies \]
\[ W = \frac{1}{2} \int d^3x \, \vec{A} \cdot \vec{J} \]

**Self- and Mutual Inductances.**

\[ \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int d^3x' \, \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} \]

\[ \implies W = \frac{\mu_0}{8\pi} \int d^3x \, d^3x' \, \frac{\vec{J}(\vec{x}) \cdot \vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} \]

If we have \( N \) circuits with currents \( I_1, \ldots, I_N \):

\[ W = \frac{1}{2} \sum_{i=1}^{N} \mathcal{L}_i I_i^2 + \frac{1}{2} \sum_{i \neq j} \mathcal{M}_{ij} I_i I_j \]

**Definition**

\( \mathcal{L}_i \sim \text{self-inductance} \)

\( \mathcal{M}_{ij} \sim \text{mutual inductance between } i \text{ and } j \)
$$W = \frac{M_o}{8\pi} \sum_{i=1}^{N} \sum_{j=1}^{N} \int \frac{d^3x_i}{V_i} \int \frac{d^3x_j}{V_j} \frac{\vec{J}(\vec{x}_i) \cdot \vec{J}(\vec{x}_j)}{|\vec{x}_i - \vec{x}_j|}$$

$$M_{ij} = \frac{M_o}{4\pi I_i I_j} \int d^3x_i \int d^3x_j \frac{\vec{J}(\vec{x}_i) \cdot \vec{J}(\vec{x}_j)}{|\vec{x}_i - \vec{x}_j|}$$

Example: self-inductance of a solenoid:

$$\nabla \times \vec{H} = \vec{J} \Rightarrow \vec{H}_m = N I \vec{z}, \vec{H}_{out} = 0$$

filled with material with permeability \(\mu\):

$$W = \frac{1}{2} \int d^3x \ \vec{B} \cdot \vec{H} = \frac{1}{2} \mu N^2 I^2 \pi R^2 L$$

$$L = \mu N^2 \pi R^2 L$$

Linear currents:

$$W = \frac{1}{2} \int d^3x \ \vec{J} \cdot \vec{A} = \frac{1}{2} \int \vec{A} \cdot d\vec{l} \cdot I$$

$$= I \cdot \frac{1}{2} \int d\alpha \cdot \nabla \times \vec{A} = \frac{1}{2} I \oint \vec{A} \cdot d\vec{l} = \frac{1}{2} N \Phi$$

$$= \frac{\Phi}{B}$$

magnetic flux

$$L = \frac{\Phi}{I}$$

formula known from undergraduate E&M
Magnetic Shielding:

\[ \nabla^2 \Phi_n = 0 \text{ everywhere} \Rightarrow \]

\[ \Phi_1 = \sum_e \left( A_e r^e \Phi_e (\cos \Theta) \right) \\ r < a \]

\[ \Phi_2 = \sum_e \left( B_e r^e + C_e r^{-e-1} \right) \Phi_e (\cos \Theta) , \quad a < r < b \]

\[ \Phi_3 = \sum_e D_e r^{-e-1} \Phi_e (\cos \Theta) , \quad r > b \]

Imposing conservation of \( H_x \) and \( B_n \) at boundaries:

\[ \frac{\partial \Phi_n}{\partial \Theta} (b^+)=\frac{\partial \Phi_n}{\partial \Theta} (b^-) \]

\[ \mu_0 \frac{\partial \Phi_1}{\partial r} (r=a) = \mu \frac{\partial \Phi_2}{\partial r} (r=a) \quad \mu \frac{\partial \Phi_2}{\partial r} (r=b) = \mu_0 \frac{\partial \Phi_3}{\partial r} (r=b) \]

Only \( l = 1 \) terms survive:

\[ D_1 = \frac{(2\mu + \mu_0)(\mu - \mu_0)}{(2\mu + \mu_0)(\mu + 2\mu_0) - \frac{2}{b^3} \frac{a^3}{b^3} (\mu - \mu_0)^2} \left( b^3 - a^3 \right) H_0 \]

\[ A_1 = -\frac{9 \mu \mu_0}{(2\mu + \mu_0)(\mu + 2\mu_0) - \frac{2}{b^3} \frac{a^3}{b^3} (\mu - \mu_0)^2} H_0 \]
For material with high magnetic permeability
\[ D_1 = \beta^3 H_0, \quad A_1 = -\frac{9\mu_0}{2\mu(1 - \frac{a^3}{b^3})} H_0 \]

\[ \Rightarrow A_1 \to 0 \text{ as } \mu \to \infty \Rightarrow \Phi_1 \to 0 \text{ as } \mu \to \infty \Rightarrow \]

\[ \Rightarrow \vec{H}, \vec{B} \to 0 \text{ inside as } \mu \to \infty \Rightarrow \]

magnetic screening!

Final: closed books, 2 cheat sheets (one from midterm, one new)
allowed 5 problems, covers the whole quarter's material, study w/ problems + midterm =) should do fine

Dec. 5, 9:30 - 11:18 a.m., Smith 1180