Example: Find magnetic dipole moment of a rotating charged sphere.

\[ \mathbf{m} = \frac{1}{2} \int d^3x \, \mathbf{x} \times \mathbf{J} \]

\[ \mathbf{J} = \rho \cdot \mathbf{V} = \rho \, \mathbf{\omega} \times \mathbf{x} \]

\[ \Rightarrow \mathbf{m} = \frac{\rho}{2} \int d^3x \, \mathbf{x} \times (\mathbf{\omega} \times \mathbf{x}) = \]

\[ = \frac{\rho}{2} \int d^3x \, \left[ \mathbf{\omega} \left| \mathbf{x} \right|^2 - \mathbf{x} \left( \mathbf{x} \cdot \mathbf{\omega} \right) \right] \]

\[ \Rightarrow \text{as } \mathbf{\omega} = \omega \mathbf{\hat{z}} \Rightarrow m_x = m_y = 0 \]

\[ \Rightarrow m_z = \frac{\rho}{2} \omega \int d^3x \left[ r^2 - z^2 \right] = \]

\[ = \frac{\rho}{2} \omega \, 2\pi \int_0^a dr \cdot r^2 \int_0^\pi \, d\cos \theta \left[ r^2 - r^2 \cos^2 \theta \right] = \]

\[ = \frac{\rho}{2} \omega \, 2\pi \frac{a^5}{5} \left[ 2 - \frac{2}{3} \right] = \pi \omega \rho a^5 \frac{4}{15} \]

\[ \Rightarrow \text{as } q = \frac{4}{3} \pi a^3 \rho \Rightarrow m = \frac{1}{5} q \omega a^2 \]

Torque on \( \mathbf{m} \): \[ \mathbf{N} = \mathbf{m} \times \mathbf{B}(0) \quad \text{(by definition of } \mathbf{B}) \]
Consider a system of localized currents in external magnetic induction $\mathbf{B}$:

$$\mathbf{F} = \int d^3x \, \mathbf{J}(x) \times \mathbf{B}(x)$$

If $\mathbf{B}$ is slowly varying, write

$$\mathbf{B}(x) = \mathbf{B}(0) + (\mathbf{x} \cdot \nabla) \mathbf{B}(0) + \ldots$$

$$\Rightarrow \mathbf{F} = \left[ \int d^3x \, \mathbf{J}(x) \right] \times \mathbf{B}(0) + \int d^3x \, \mathbf{J}(x) \times (\mathbf{x} \cdot \nabla) \mathbf{B}(0)$$

$$\Rightarrow \mathbf{F}_i = \int d^3x \, \varepsilon_{ijk} \mathbf{J}_j(x) \cdot (\mathbf{x} \cdot \nabla) B_k(0) = \int d^3x \, \varepsilon_{ijk} \mathbf{J}_j(x) \cdot (\partial_k B_k)\bigg|_{x=0} = \left(\partial e B_k\right)\bigg|_{x=0} \varepsilon_{ijk} \int d^3x \, x_i \mathbf{J}_j$$

$$\Rightarrow \varepsilon_{ijk} \int d^3x \, (x_i \mathbf{J}_j + x_j \mathbf{J}_i) = 0 \Rightarrow \mathbf{F}_i = (\partial e B_k)\bigg|_{x=0} \varepsilon_{ijk} \int d^3x \, \frac{1}{2} \left[ x_e \mathbf{J}_d - x_d \mathbf{J}_e \right]$$

$$\Rightarrow \mathbf{F}_i = \varepsilon_{ijk} \int d^3x \, e e_{12} \left[ x_e \mathbf{J}_d - x_d \mathbf{J}_e \right]$$
\[ m_i = \frac{1}{2} \varepsilon_{ijk} \epsilon_{d'k'} \int d^3x \times \partial_i \mathbf{J}_d = \]

\[ \Rightarrow \varepsilon_{ijm} \cdot m_n = \frac{1}{2} \varepsilon_{ijm} \varepsilon_{d'k'} \int d^3x \times \partial_i \mathbf{J}_d = \]

\[ = \frac{1}{2} \int d^3x \left[ \frac{\mathbf{\varepsilon} \times \mathbf{\mathbf{J}}_d}{\sigma} - X_d \cdot \mathbf{J_e} \right] \]

\[ \Rightarrow F_i = \left( \partial_e B_e \right) \bigg|_{x=0} \varepsilon_{ijm} \varepsilon_{d'k'} m_n = \]

\[ = \left( \partial_e B_e \right) \bigg|_{x=0} m_n \cdot \left[ \delta_{i} \varepsilon_{d'k'} - \delta_{d'} \varepsilon_{i} \mathbf{J}_e \right] = \]

\[ = m_k \left( \partial_i B_e \right) \bigg|_{x=0} - m_i \left( \partial_k B_e \right) \bigg|_{x=0} \]

\[ \Rightarrow \nabla \cdot \mathbf{F} = 0 \]

\[ \mathbf{F} = \nabla \left( \mathbf{m} \cdot \mathbf{B} \right) \]

If \( \mathbf{F} = -\nabla U \), with \( U \) the potential energy, \( \Rightarrow \)

\[ U = -\mathbf{m} \cdot \mathbf{B} \]

Tends to align dipoles with the magnetic induction \( \mathbf{B} \).
Macroscopic Equations of Magnetostatics

Similar to electrostatics, let's divide the currents into "free" and "bound".

Bound currents are due to magnetic moments of molecules/atoms in the medium and are described by magnetization \( \vec{M}(x') \).

The resulting vector potential is

\[
\vec{A}(x) = \frac{\mu_0}{4\pi} \int d^3x' \left\{ \frac{\vec{J}(x')}{|x' - x'|} + \frac{\vec{M}(x') \times (x - x')}{|x' - x'|^3} \right\}
\]

free current magnetization/bound currents

The 2nd term is

\[
\int d^3x' \frac{\vec{M}(x') \times (x - x')}{|x' - x'|^3} = \int d^3x' \frac{\vec{M}(x') \times \nabla' \frac{1}{|x' - x'|}}{|x' - x'|} = \text{(parts)} = \int d^3x' \frac{1}{|x' - x'|} \nabla' \times \vec{M}(x'), \text{ such that}
\]

\[
\vec{A}(x) = \frac{\mu_0}{4\pi} \int d^3x' \frac{\vec{J}(x') + \nabla' \times \vec{M}(x')}{|x' - x'|}
\]

\[\Rightarrow\] effective current density due to \( \vec{M} \) is

\[\vec{J}_M = \nabla \times \vec{M}\]