\[ W = \frac{1}{2} \frac{Q^2}{\varepsilon} \text{ Volume} \]

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Capacitance:
\[ C = \frac{Q}{\Delta V} = \frac{Q}{E \cdot d} = \frac{Q}{\frac{Q}{\varepsilon} \cdot d} = \frac{\varepsilon}{d} \]

\[ \Rightarrow \frac{C}{\frac{\varepsilon}{d}} = \frac{\varepsilon}{d} \]

\[ \text{in vacuum } \varepsilon = \varepsilon_0 \Rightarrow \frac{C}{\frac{\varepsilon}{d}} = \frac{\varepsilon_0}{d} \]

\[ \text{works!} \]

Forces:
\[ F_y = -\left( \frac{\partial W}{\partial y} \right)_q \]

Force due to displacement in \( y \)-direction with sources \( Q \) fixed (insulated from external world)

Example:

\[ \begin{align*}
\varepsilon & \quad + Q \\
\hline \\
\hline \\
\hline \\
\hline
- Q
\end{align*} \]

\[ L \times L \text{ square plates.} \]

In general, surface charge density is different in vacuum \& dielectric parts:

\[ \sigma_d = \varepsilon E_d \quad \sigma_v = \varepsilon_0 E_v \]
At the interface $E_{d, t} = E_{v, t} \Rightarrow E_d = E_v = E$

$\Rightarrow Q = L y \sigma_d + L (L - y) \sigma_v = L (y \sigma + (L - y) \sigma_0) E$

$\Rightarrow E = \frac{Q}{L [\varepsilon_y + \varepsilon_0 (L - y)]}$

In dielectric $D = \varepsilon E$, in vacuum $D = \varepsilon_0 E$

$\Rightarrow$ total energy $W = \frac{1}{2} \int dy L (D_d \cdot E_d +$

$+ \frac{1}{2} d (L - y) L \cdot D_v \cdot E_v = \frac{1}{2} \int dy L \cdot \varepsilon \cdot \left( \frac{Q}{L [\varepsilon_y + \varepsilon_0 (L - y)]} \right)^2 +$

$+ \frac{1}{2} d (L - y) L \cdot \varepsilon_0 \left( \frac{Q}{L [\varepsilon_y + \varepsilon_0 (L - y)]} \right)^2 = \frac{1}{2} d \frac{Q^2}{L [\varepsilon_y + \varepsilon_0 (L - y)]} \Rightarrow F = - \left( \frac{\partial W}{\partial y} \right)_Q = 0$

$\Rightarrow \left( \frac{d \sigma^2 (L - \varepsilon_0)}{L [\varepsilon_y + \varepsilon_0 (L - y)]^2} \right) = F > 0$

$F > 0 \Rightarrow$ the force pulls the slab inside the capacitor

$\varepsilon = \varepsilon_0 \Rightarrow F = 0$ no force in vacuum.
The problem is different if capacitor plates are held at constant potential difference \( V \).

\[
V = E \cdot d \implies E = \frac{V}{d}
\]

\[
W = \frac{1}{2} L d \cdot E^2 \left[ \varepsilon y + \varepsilon_0 (\gamma - y) \right] = \frac{1}{2} L d \cdot \frac{1}{2} \left[ \varepsilon y + \varepsilon_0 (\gamma - y) \right] = \frac{1}{2} \frac{L U^2}{d} \left[ \varepsilon y + \varepsilon_0 (\gamma - y) \right]
\]

\[
\Rightarrow F = \left( \frac{\partial W}{\partial y} \right) V = \frac{1}{2} \frac{L U^2}{d} (\varepsilon - \varepsilon_0) > 0
\]

The system is not isolated anymore.

When we move the dielectric, we first fix the charges:

\[
\Rightarrow \delta W_1 = \frac{1}{2} \int \delta \Phi_1 \, d^3x
\]

Then we let the charges exit/enter the system to keep potential constant:

\[
\delta W_2 = \frac{1}{2} \int d^3x \left[ \delta \Phi_2 + \Phi_0 \delta \Phi_2 \right]
\]

Now, to keep \( \Phi \) constant, we need \( \delta \Phi_1 = -\delta \Phi_2 \), \( \Rightarrow \delta W_2 = -\delta W_1 + \frac{1}{2} \int d^3x \Phi \delta \Phi_2 \). Now, \( \varepsilon_0 \Phi_0 = -\frac{\Phi}{\varepsilon} \) \( \Rightarrow \)

\( \Rightarrow \) both terms in \( \delta W_2 \) are equal \( \Rightarrow \delta W_2 = -2 \delta W_1 \), \( \Rightarrow \delta W_1 = \delta W_1 + \delta W_2 = -\delta W_1 = -\delta W_0 \Rightarrow \)

\[
F = \left( \frac{\partial W}{\partial \gamma} \right)_V
\]