Monochromatic Plane Wave

Energy density:
\[ u = \frac{1}{2} (\vec{E} \cdot \vec{E} + \vec{B} \cdot \vec{H}) \]

Pointing vector:
\[ \vec{\Phi} = \vec{E} \times \vec{H} \]

Maxwell stress tensor:
\[ 6 \varepsilon \vec{i} \vec{j} = \varepsilon \vec{E} \varepsilon \vec{E} + \mu \vec{H} \vec{H} \]
\[ -\frac{1}{2} \varepsilon \vec{i} \vec{j} (\varepsilon \vec{E}^2 + \mu \vec{H}^2) \]

(for LiH media)

\[
\begin{cases}
\vec{E} = E_0 \cos(\omega t - \vec{ka}) \\
\vec{B} = B_0 \cos(\omega t - \vec{ka})
\end{cases}
\]

\[ \langle u \rangle = \frac{1}{2} \varepsilon E_0^2 \]
\[ \langle \vec{\Phi} \rangle = \frac{1}{2} \sqrt{\frac{\varepsilon}{\mu}} E_0^2 \hat{k} = \langle u \rangle \vec{v} \]

\( \vec{v} \) energy flow

where \( \vec{v} = \frac{\hat{i}}{\sqrt{\varepsilon \mu}} \)

\( \langle \ldots \rangle \) time averaged
Polarization

\[ \mathbf{E} = \text{Re} \left\{ E_0 e^{-i \omega t + i \mathbf{k} \cdot \mathbf{r}} \right\} \text{ ~ plane wave} \]

choose a basis:

\[ \hat{E}_0 = E_1 \hat{\xi}_1 + E_2 \hat{\xi}_2 \]

\( E_1 \) and \( E_2 \) are complex in general (may contain phase shifts)

\( \hat{\xi}_1, \hat{\xi}_2 \) are linear polarizations

\[ \text{Def.} \quad \hat{\xi}_t = \frac{1}{\sqrt{2}} (\hat{\xi}_1 + i \hat{\xi}_2) \text{ ~ circular polarizations} \]

\( \text{Def.} \quad \hat{\xi}_l = \frac{1}{\sqrt{2}} (\hat{\xi}_1 - i \hat{\xi}_2) \text{ ~ left circular polarizations} \)

\[ \Rightarrow \hat{E}_0 = E_1 \hat{\xi}_1 + E_2 \hat{\xi}_2 + E_3 \hat{\xi}_3 \]

\( \)
Reflection and Refraction

Incident wave:
\[
\begin{align*}
\vec{E} &= E_0 \ e^{i(\vec{k} \cdot \vec{x} - \omega t)} \\
\vec{B} &= \frac{1}{\omega} \ \vec{k} \times \vec{E} = \frac{\mu \varepsilon}{k} \ \vec{k} \times \vec{E}
\end{align*}
\]

Refracted wave:
\[
\begin{align*}
\vec{E}' &= E_0' \ e^{i(\vec{k}' \cdot \vec{x} - \omega't)} \\
\vec{B}' &= \frac{\mu_1 \varepsilon}{k'} \ \vec{k}' \times \vec{E}'
\end{align*}
\]

Reflected wave:
\[
\begin{align*}
\vec{E}'' &= E_0'' \ e^{i(\vec{k}'' \cdot \vec{x} - \omega''t)} \\
\vec{B}'' &= \frac{\mu_2 \varepsilon}{k''} \ \vec{k}'' \times \vec{E}''
\end{align*}
\]

Match boundary conditions: \(\nabla \cdot \vec{B} = 0 \Rightarrow B_n\) is zero,
\(\nabla \cdot \vec{B}' = 0 \Rightarrow B_n\) is continuous,
\(\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \Rightarrow \nabla \times \vec{E} - i\omega \vec{B} = 0 \Rightarrow E_t\) is continuous as \(\vec{B}\) has no \(\delta\)-function singularity at \(z = 0\).
\(\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} = -i\omega \vec{D} \Rightarrow H_t\) is continuous (same reason) at \(z = 0\).

To have any boundary conditions need
\[\omega = \omega' = \omega''\Rightarrow k = k'' = \sqrt{\mu \varepsilon} \omega, \ k' = \sqrt{\mu_1 \varepsilon} \omega\]

Spatial phase factors should also be equal at \(z = 0\):
\[\frac{\vec{k} \cdot \vec{x}}{\sqrt{\mu \varepsilon}} = \frac{\vec{k}' \cdot \vec{x}}{\sqrt{\mu_1 \varepsilon}} = \frac{\vec{k}'' \cdot \vec{x}}{\sqrt{\mu_2 \varepsilon}}, \ \forall x, y, z = 0\]
Choose \( \mathbf{w} = (w_x, 0, w_z) \Rightarrow \mathbf{w} \cdot \mathbf{x} |_{x=0} = k_x x \Rightarrow \text{no y-dependence} \)

\( \Rightarrow \) there should be no \( y \)-dependence in \( \mathbf{k}' \cdot \mathbf{x} \) and in \( \mathbf{k}'' \cdot \mathbf{x} \)

as well \( \Rightarrow k_y = h_y = 0 \Rightarrow \) all lie in the same plane

\( k \cdot \sin \Theta = k' \cdot \sin \Theta' = k'' \cdot \sin \Theta'' \)

\( \Rightarrow \) as \( k = k'' \Rightarrow \boxed{\Theta = \Theta''} \) the angle of reflection is equal to angle of incidence!

as \( k = \sqrt{\mu_3} \omega \) and \( k' = \sqrt{\mu_3'} \omega \)

\( \sqrt{\mu_3} \sin \Theta = \sqrt{\mu_3'} \sin \Theta' \). Remember \( n = \sqrt{\mu_3} \) (index of refraction) \( \Rightarrow \boxed{n \sin \Theta = n' \sin \Theta'} \).

Snell’s Law!

The only thing left is to find \( \vec{E}_0' \) & \( \vec{E}_0'' \) using 6 eqns:

\( \text{D}_{\mu} \) continuous \( \Rightarrow \hat{n} \cdot \left[ \varepsilon \left( \vec{E}_0 + \vec{E}_0'' \right) - \varepsilon' \vec{E}_0' \right] = 0 \)

\( \text{B}_{\mu} \) continuous \( \Rightarrow \hat{n} \cdot \left[ k \vec{E}_0 + k'' \vec{E}_0'' - k' \vec{E}_0' \right] = 0 \)

\( (\text{and } \omega = \omega' = \omega'') \)

\( E_\theta \) continuous \( \Rightarrow \hat{n} \times \left[ \vec{E}_0 + \vec{E}_0'' - \vec{E}_0' \right] = 0 \)

\( H_\theta \) continuous \( \Rightarrow \left[ \frac{1}{\mu} \left( \vec{k} \times \vec{E}_0 + k'' \vec{E}_0'' \right) - \frac{1}{\mu'} \left( k' \vec{E}_0' \right) \right] \times \hat{n} = 0 \)

\( \vec{E}_0', \vec{E}_0'', \varepsilon_0 \) 6 unknowns while there are 6 equations above.

Consider 2 cases (linear polarization):

\( \boxed{\vec{E}_0' \perp \text{plane of incidence}} \)

\( \vec{E}_0, \vec{E}_0', \vec{E}_0'' \parallel \hat{y} \)
$E_0 + E_0'' - E_0' = 0$

$\frac{1}{\mu} \left( \frac{\partial}{\partial t} (k E_0 \cos \theta - k'' E_0'' \cos \phi'') - \frac{1}{\mu} \right) k' E_0' \cos \theta' = 0$

$\Rightarrow k = k'' = \sqrt{\mu \varepsilon_0} \omega$, $k' = \sqrt{\mu \varepsilon_1} \omega$ $\Rightarrow$ and $\theta = \theta''$

$E_0 + E_0'' - E_0' = 0$

$\sqrt{\frac{\varepsilon}{\mu} (E_0 - E_0'')} \cos \theta - \sqrt{\frac{\varepsilon_1}{\mu_1}} E_0' \cdot \cos \theta' = 0$

$1st \text{ eqn. } 0 = 0$ \hspace{1cm} $2nd \text{ eqn. } k E_0 \sin \theta + k'' E_0'' \sin \phi' - k' E_0' \sin \theta' = 0 \Rightarrow (E_0 + E_0'') \sin \theta - \sqrt{\varepsilon_1 \varepsilon_0} E_0' \sin \theta' = 0$

$\Rightarrow \sqrt{\varepsilon_1 \varepsilon_0} \sin \theta = \sqrt{\varepsilon_1 \varepsilon_0} \sin \theta' \text{ (Snell's law)} \Rightarrow E_0 + E_0'' - E_0' = 0$

$\Rightarrow$ duplicated the 3rd one.

Using Snell's law ($n \sin \theta = n' \sin \theta'$) to get rid of $n'$, we write (work it out yourself):

\[ \frac{E_0'}{E_0} = \frac{2n \cos \theta}{n \cos \theta + \frac{\lambda}{\mu_1} \sqrt{n'^2 - n^2 \sin^2 \theta}} \]

\[ n = \sqrt{\frac{\lambda}{\mu_0 \varepsilon_0}} \]

\[ n' = \sqrt{\frac{\lambda}{\mu_0 \varepsilon_0}} \]

\[ \text{Fresnel equations} \]
2 independent equations (Snell's law):

\[
\begin{cases}
(E_0 + E''_0) \cos \theta - E'_0 \cos \theta' = 0 \\
\sqrt{\frac{\varepsilon}{\varepsilon'}} (E_0 - E''_0) - \sqrt{\frac{\varepsilon'}{\varepsilon}} E'_0 = 0
\end{cases}
\]

(others two can be reduced to these)

Solve:

Using Snell's law

\[
\begin{align*}
\frac{E'_0}{E_0} &= \frac{2 \mu' \mu \cos \theta}{\mu' n^2 \cos \Theta + n \sqrt{n^2 - n^2 \sin^2 \Theta}} \\
\frac{E''_0}{E_0} &= \frac{-\mu \mu' \cos \Theta + n \sqrt{n^2 - n^2 \sin^2 \Theta}}{\mu' n^2 \cos \Theta + n \sqrt{n^2 - n^2 \sin^2 \Theta}}
\end{align*}
\]

Normal incidence: \( \Theta = 0 \) \( \Rightarrow \) both I and II give

\[
\begin{align*}
\frac{E'_0}{E_0} &= \frac{2n}{n + \frac{\mu}{\mu'}} \\
\frac{E''_0}{E_0} &= \frac{n - \frac{\mu}{\mu'}}{n + \frac{\mu}{\mu'}}
\end{align*}
\]

Polarization by reflection: put \( \mu = \mu' \) for simplicity.

I: \( \frac{E'_0}{E_0} = \frac{n \cos \theta - \sqrt{n^2 - n^2 \sin^2 \theta}}{n \cos \theta + \sqrt{n^2 - n^2 \sin^2 \theta}} \)

\( \Rightarrow \) different =)

II: \( \frac{E''_0}{E_0} = \frac{-n^2 \cos \theta + n \sqrt{n^2 - n^2 \sin^2 \theta}}{n^2 \cos \theta + n \sqrt{n^2 - n^2 \sin^2 \theta}} \)

\( \Rightarrow \) reflected light is polarized!