Physics 880.06: Problem Set 3

Note: please turn these problems into the mailbox of the grader, Wissam Al-Saidi, by 12 noon on Friday, April 26. Each problem is worth 10 points.

1. The “Bogoliubon” operators were introduced in class by the transformation

\[ c_k^\dagger = u_k \gamma_{k,0} + v_k \gamma_{k,1}, \]
\[ c_{-k}^\dagger = -v_k^* \gamma_{k,0} + u_k \gamma_{k,1}, \] (1)

where the coefficients \( u_k \) and \( v_k \) satisfy the normalization condition

\[ |u_k|^2 + |v_k|^2 = 1. \] (3)

Show that the operators \( \gamma_{k,0} \), \( \gamma_{k,1} \) and their Hermitian conjugates satisfy the standard Fermi anticommutation relations

\[ [\gamma_{k,i}, \gamma_{k,j}]_+ = \delta_{ij}, \] (4)

where \( i \) and \( j \) can take on the values 1 and 2, and \([..., ...]_+ \) denotes an anticommutator.

2. The operator \( a_{k}^\dagger = c_{k1}^\dagger c_{-k1} \) creates a pair of electrons with opposite wave vector and opposite spin.

Show that if \( k \neq k' \), the commutator (not the anticommutator) \([a_k^\dagger, a_{k'}] = 0\), as expected for Bose operators, but that if \( k = k' \), this commutator does not equal unity, as one would expect for Bose operators, and calculate \([a_k^\dagger, a_k]\).

3. The electronic specific heat of the Bogoliubons is, as noted by Tinkham,

\[ C_{es} = T \frac{dS_{es}}{dT}, \] (5)

where \( S_{es} \) is the entropy of the Bogoliubons and is given by the standard result for a collection of Fermions:

\[ S = -2k_B \sum_k [(1 - f_k)ln(1 - f_k) + f_k ln f_k]. \] (6)
Here
\[ f_k = \frac{1}{e^{\beta E_k} + 1} \] (7)
and
\[ E_k = \sqrt{\Delta^2 + \xi_k^2} \] (8)
\(\Delta\) being the energy gap, which we assume is real, and \(\xi_k = \epsilon_k - E_F\).

Show that the specific heat can be written as
\[ C_{es} = 2\beta k_B \sum_k \left( -\frac{\partial f_k}{\partial E_k} \right) \left( E_k^2 + \frac{1}{2} \frac{\beta d\Delta^2}{d\beta} \right) , \] (9)
where \(\beta = 1/(k_B T)\).

Show explicitly that, if \(\Delta\) is temperature-independent, the specific heat varies at low temperatures as \(\exp(-\Delta/k_B T)\) multiplied by a function which varies more slowly with temperature. (This is also true even if \(\Delta\) is dependent on temperature, provided that it remains finite at \(T = 0\).)