1. Consider the antiferromagnetic Heisenberg model

\[ H = +J \sum_{\langle ij \rangle} S_i \cdot S_j \]  

where \( S_i \) is a spin-1/2 quantum spin operator, the sum runs over distinct nearest neighbor pairs as discussed in class, and \( J > 0 \). Assume that the spins lie on a lattice which can be divided into two sublattices, such that all the nearest neighbors of spins on one sublattice are spins on the other sublattice. Examples of such “bipartite” lattices include the square lattice in two dimensions, and the body-centered cubic lattice in three dimensions.

Prove that the antiferromagnetic state, in which all the spins are up on one sublattice and down on the other, is not an eigenstate of the antiferromagnetic Heisenberg model.

2. (20 pts.)

In class we introduced the operator

\[ S_{k,+} = \frac{1}{\sqrt{N}} \sum_{\ell} e^{i k \cdot R_{\ell}} S_{\ell,+}, \]  

where \( S_{\ell,+} \equiv S_{\ell,x} + i S_{\ell,y} \). Here \( N \) is the number of sites on the lattice.

(a). Show that the hermitean conjugate of \( S_{k,+} \), denoted \( S_{k,-} \) is given by

\[ S_{k,-} = \frac{1}{\sqrt{N}} \sum_{\ell} e^{-i k \cdot R_{\ell}} S_{\ell,-}. \]  

(b). Show that, if we approximate the operator \( S_{\ell,z} \) by \(-1/2\) (as is true in the ground state of the spin-1/2 Heisenberg model), then

\[ [S_{k,-}, S_{k',+}] = \frac{1}{N} \sum_{\ell} e^{i(k'-k) \cdot R_{\ell}}. \]
As shown in fall quarter, the sum on the right-hand side simplifies if we consider only \( k \) and \( k' \) which satisfy Born-Von Karman boundary conditions (see Ashcroft and Mermin, p. 135). These are basically a generalization of periodic boundary conditions to a crystal. In that case, the sum on the right-hand side of this equation equals the Kronecker delta function \( \delta_{k,k'} \). The commutator then becomes

\[
[S_{k,-}, S_{k',+}] = \delta_{k,k'}.
\]  

These are the same commutation relations as those for the raising and lowering operators of harmonic oscillators as discussed last quarter.

(c). Show also that \([S_{k,+}, S_{k',+}] = [S_{k,-}, S_{k',-}] = 0\) in the same approximation (where \( S_{\ell,z} \sim -1/2 \)).

(d). Show that the inverse transformation which defines \( S_{\ell,+} \) in terms of \( S_{k,+} \) is

\[
S_{\ell,+} = \frac{1}{\sqrt{N}} \sum_k e^{-ik\cdot R_\ell} S_{k,+},
\]

where the sum on \( k \) runs over the first Brillouin zone. Hint: you may assume the relation

\[
\frac{1}{N} \sum_k e^{ik(R_\ell-R_{\ell'})} = \delta_{\ell,\ell'}.
\]

Analogous relations hold for \( S_{\ell,-} \) and for \( S_{\ell,z} \).

(d). OPTIONAL; NOT TO BE TURNED IN. Hence, show that, if we make the approximation \( S_{\ell,z} \sim -1/2 \), the Heisenberg Hamiltonian can be written in the approximate form

\[
H = E_g + \sum_k \epsilon_k S_{k,+} S_{k,-},
\]

where the sum runs over the first Brillouin zone. This result, together with the commutation relations proved earlier in this problem, shows that \( H \) can be written as a ground state energy plus a sum of independent harmonic oscillator Hamiltonians.
3. In class, it was stated that the expectation value of the number of excitations \( \langle n_k \rangle \) in the \( k^{th} \) state equals

\[
\langle n_k \rangle = \frac{\sum_{n=0}^{\infty} n \exp(-\beta n \epsilon_k)}{\sum_{n=0}^{\infty} \exp(-\beta n \epsilon_k)},
\]

where \( \beta = 1/(k_B T) \). Show that this expression reduces to

\[
\langle n_k \rangle = \frac{1}{e^{\beta \epsilon_k} - 1}.
\]

4. As stated in class, the internal energy of a Heisenberg ferromagnet, in the spin-wave approximation, is

\[
U \sim E_g + \sum_k \epsilon_k \langle n_k \rangle,
\]

where \( \langle n_k \rangle \) is the number of spin waves in state \( k \) as given in the previous problem, and the sum runs over the first Brillouin zone.

Show that, in three dimensions, at sufficiently low temperatures, this internal energy is given approximately by

\[
U \sim E_g + KT^{5/2}
\]

and find the constant \( K \) in terms of the spin wave stiffness constant \( D \). You may express your answer in terms of a numerical integral, if necessary; in that case, do not evaluate the numerical integral.

Hint: you may need to use the relation (discussed last quarter) for turning a sum over \( k \) space into an integral in three dimensions:

\[
\sum_k \rightarrow \frac{V}{(2\pi)^3} \int d^3k.
\]