1. (10 points). Spin-spin correlation function for the Ising model in 1d. Consider the ferromagnetic Ising model in 1D with zero applied magnetic field. The Hamiltonian is

\[ H = -J \sum_{n=1}^{N-1} S_n S_{n+1}, \]  

where \( S_n \) takes on the values ±1, \( J > 0 \), and we assume FREE boundary conditions, so that \( S_1 \) and \( S_N \) are not attached to any neighbors.

(a) The partition function for this special case of \( B = 0 \) can be easily calculated by defining new variables \( V_n = S_n S_{n+1} \), with \( n \) running from 1 to \( N - 1 \) These \( V_n \)'s also take on the allowed values ±1 and are independent of one another. Hence, the partition function can be written

\[ Q_N(T) = 2 \sum_{V_1 = \pm 1} \cdots \sum_{V_{N-1} = \pm 1} \exp(-H/k_BT), \]  

where \( H = -J \sum_{n=1}^{N-1} V_n \). The factor of 2 in front of the partition function allows for two possible orientations of the spin \( S_1 \). Since \( H \) is now the sum of independent variables, the partition function can be easily calculated. Use this simple transformation to calculate the partition function for a chain of \( N \) spins.

(b). Calculate the correlation function \( \langle S_p S_q \rangle \), where \( q > p \) but \( |q - p| \ll N \). (\( p \) and \( q \) are site indices.) Hint: write \( S_p S_q = (S_p S_{p+1})(S_{p+1} S_{p+2})\ldots(S_{q-1} S_q) \) and use the change of variables described in (a). Why is it possible to write \( S_p S_q \) in this way?

(c). Show that, for \( k_BT \ll J \), \( \langle S_p S_q \rangle \sim \exp[ -|p - q|/\xi(T) ] \), where the correlation length \( \xi(T) \sim \exp(2J/k_BT)/2 \).

2. (20 points) Mean-field theory for the XY model: A mean-field theory for the XY model can be obtained by analogy with that of the
Ising model. To find the mean-field properties, consider the XY model Hamiltonian

\[ H = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) - B \sum_i \cos \theta_i, \]  

(2)

where \( B \) is the applied magnetic field, assumed to be in the \( x \) direction. \( i \) and \( j \) are site indices, and the first sum runs overall distinct pairs of nearest neighbor spins. We take the order parameter \( \eta \) to be the magnetization per spin, i.e.

\[ \eta = \langle \cos \theta_i \rangle, \]  

(3)

where \( \langle ... \rangle \) denotes an average in the canonical ensemble. Note that because the coefficients \( J \) and \( B \) are independent of site, \( \eta \) should be independent of \( i \). The exact expression for \( \eta \) in the canonical ensemble is

\[ \eta = \frac{1}{Q_N} \int_0^{2\pi} d\theta_1 \ldots \int_0^{2\pi} d\theta_N \exp(-H/k_BT) \cos \theta_i, \]  

(4)

where

\[ Q_N = \int_0^{2\pi} d\theta_1 \ldots \int_0^{2\pi} d\theta_N \exp(-H/k_BT). \]  

(5)

To obtain the critical temperature in the mean-field approximation, we assume that the \( i^{th} \) spin is moving in a field equal to the applied field plus the mean-field of its \( z \) nearest neighbors. The interaction energy between the \( i^{th} \) and \( j^{th} \) spin is

\[ -J \cos(\theta_i - \theta_j) = -J [\cos \theta_i \cos \theta_j + \sin \theta_i \sin \theta_j]. \]  

(6)

We approximate the right-hand side of this equation as

\[ -J [\cos \theta_i \langle \cos \theta_j \rangle + \sin \theta_i \langle \sin \theta_j \rangle] = -J \eta \cos \theta_j, \]  

(7)

where the last equality comes from the fact that \( \langle \sin \theta_j \rangle = 0 \). Thus, in the mean-field approximation, \( \eta \) is given by the self-consistent equation

\[ \eta = \frac{\int_0^{2\pi} \exp(-H_{MF}/k_BT) \cos \theta_i d\theta_i}{\int_0^{2\pi} \exp(-H_{MF}/k_BT) d\theta_i}, \]  

(8)
where

\[ H_{MF} = -(B + zJ\eta) \cos \theta_i. \]  \hfill (9)

(a). Near the transition temperature \( T_c \), \( \eta \) is expected to be small. Expand both the numerator and the denominator in powers of \( \eta \), and find all terms through \( \eta^3 \) in the numerator and through \( \eta^2 \) in the denominator. Thus, obtain all terms in the ratio through \( \eta^3 \).

(b). Solve the resulting cubic equation for \( B = 0 \), and show that there exists a \( T_c \) such that this equation has real, non-zero solutions for \( \eta \) for \( T < T_c \) but the only real solution for \( T > T_c \) is \( \eta = 0 \). Find \( T_c \) in terms of \( z \) and \( J \). Show that for \( T < T_c \), \( |\eta| \propto (T - T_c)^\beta \), and find the exponent \( \beta \).

(c). For \( T > T_c \) and \( B \neq 0 \), find \( \eta \) to first order in \( B \). Show that \( (\partial \eta/\partial B)_{B=0} \propto (T - T_c)^{-\gamma} \) and find \( \gamma \).

Note that this solution fails badly in \( d = 1 \) and \( d = 2 \) because, as shown in class, the spontaneous magnetization in both cases vanishes for \( T > 0 \).

3. (20 pts.) **Properties of Spin Operators**: In this problem, you will prove some of the properties of spin operators stated in class, starting with the commutation relations \([S_x, S_y] = iS_z, S_y, S_z] = iS_x\), and \([S_z, S_x] = iS_y\). (Note: in this convention, the spin angular momentum operator is \( \hbar \) multiplied by the spin operator.) Define the raising and lowering operators \( S_+ = S_x + iS_y \) and \( S_- = S_x - iS_y \), and the squared total spin angular momentum operator \( S^2 = S_x^2 + S_y^2 + S_z^2 \). Prove the following:

(a). \( [S_z, S^2] = 0 \).

(b). \( [S_z, S_\pm] = \pm S_\pm \).

(c). \( [S_+, S_-] = 2S_z \).

(d). \( S_-S_+ = S^2 - S_z^2 - S_z \) and \( S_+S_- = S^2 - S_z^2 + S_z \).

(e). Since \( [S_z, S^2] = 0 \), we can find a complete set of states which are simultaneously eigenstate of \( S^2 \) and \( S_z \). Denote these eigenstates as \( |Sm\rangle \). Let the eigenvalue of \( S^2 \) be denoted \( S(S + 1) \) and that of \( S_z \) be denoted \( m \). In this part, you will show that \( S \) is an integer or a half-integer, and that \(-m \leq S \leq +m\).
(i). Since $S^2$ is the sum of three positive operators, its eigenvalue must be at least 0. Therefore, $S(S + 1) \geq 0$. Use the same argument to show that the $\langle Sm|S_+S_-|Sm\rangle \geq 0$, and therefore that $S(S+1) - m^2 - m \geq 0$. Show that this is equivalent to $(S + 1/2)^2 \geq (m - 1/2)^2$.

(ii) Use a similar argument to show that $S_+^{1/2} \geq (m + 1/2)^2$.

(iii). Show that (i) and (ii) imply that $-S \leq m \leq +S$.

(iv). Use the result of part (a) to show that $S_+^{1/2} \geq (m + 1/2)^2$.

(v). Hence, show from (c) above that $\langle Sm|S_+S_-|Sm\rangle = |c|^2 \langle S, m - 1|S, m - 1\rangle = [S(S + 1) - m(m - 1)]$. Thus, if we choose the phase of $|S, m-1\rangle$ so that $c$ is a real, positive constant, then you have shown that $S_-|Sm\rangle = \sqrt{S(S + 1) - m(m - 1)}|S, m - 1\rangle$. By a similar argument, it can be shown that $S_+|Sm\rangle = \sqrt{S(S + 1) - m(m - 1)}|S, m + 1\rangle$.

Hence, it appears that if $m$ is an eigenvalue of $S_z$, then so are $m+1$, $m+2$, etc. But you have already shown that $-S \leq m \leq +S$. Therefore, for some $m$ of that does not exceed $S$, the coefficient $\sqrt{S(S + 1) - m(m + 1)}$ must vanish, in order for the series of eigenstates to terminate with $m \leq S$. This vanishing clearly occurs for $m = S$. Thus, we must have $m = S$, $S - 1$, $S - 2$, etc. A similar argument shows that the lowest $m$ value satisfies $\sqrt{S(S + 1) - m(m - 1)} = 0$ or $m = -S$. Thus $m$ must also satisfy $m = -S$, $-S + 1$, $-S + 2$, etc. Both of these conditions can be satisfied only if $S$ is an integer or a half-integer, in which case $m = -S$, $-S + 1$, $-S + 2$, etc. $S - 1$, $S$, as was to be proved.

Note: the last two paragraphs are purely explanatory, and do not ask you to do anything.