Isotopic spin - G parity

Isotopic spin was introduced in 1930's by Heisenberg to describe the approximate charge independence of the strong interactions.

\[ \gamma_N(x) = \begin{pmatrix} \gamma_p(x) \\ \gamma_n(x) \end{pmatrix} \]  

Proton field  
Neutron field

Phenomenological description valid for length scales \( \geq 1 \text{ fm} \).

U2 symmetry

Define transformation

\[ \gamma \rightarrow \gamma' = U \gamma \]

where \( U \) is a complex 2x2 matrix.
Note: if define \( \psi_1 = \psi_p, \psi_2 = \psi_n \) components of \( \psi_N \), they satisfy the canonical anti-commutation relations

\[
\{ \psi_i(x, t), \psi_j(x', t') \} = \delta_{ij} \delta^3(x-x')
\]

In order for the transformation to preserve the commutation relations we have

\[
\{ \psi'_i(x, t), \psi'_j(x', t') \} = \delta_{ij} \delta^3(x-x')
\]

\[
= u_{ik} u^+_{lj} \left\{ \psi_k(x, t), \psi_l(x', t') \right\} \delta_{ek} \delta^3(x-x')
\]

\[
\Rightarrow (uu^+)^{ij} = \delta_{ij} \text{ or } |uu^+| = 1.
\]

\[
\Rightarrow u \text{ is unitary } 2 \times 2 \text{ matrix.}
\]
The set of all complex $2 \times 2$ unitary matrices form a group under matrix multiplication. This group is called $U_2$.

Let's define the transformation in terms of operators in Fock space. There exists a unitary operator $U$ that acts on Fock space such that (let $\Psi = \Psi_\infty$)

$$U \Psi(x) U^+ \equiv u \Psi(x).$$

(We shall construct it later.)

Given

$$\Psi^\rho(x) = \int \frac{d^3k}{(2\pi)^3} \sum_s \left( b^\rho(k^-s) u(k^-,s) e^{-ik \cdot x} + d^\rho_\uparrow(k^-,s) v(k^-,s) e^{ik \cdot x} \right),$$

$$\Psi^\Lambda(x) = \begin{pmatrix} b_\Lambda(k^-,s) \\ d_\Lambda^\dagger(k^-,s) \end{pmatrix},$$

$$\left( b_\Lambda(k^-,s) \\ d_\Lambda^\dagger(k^-,s) \right).$$
\[ \Rightarrow \mathbf{U} \begin{pmatrix} b_p(\mathbf{K}, s) \\ b_n(\mathbf{K}, s) \end{pmatrix} \mathbf{U}^+ = \mathbf{u} \begin{pmatrix} b_p(\mathbf{K}, s) \\ b_n(\mathbf{K}, s) \end{pmatrix} \]

\[ \Rightarrow \mathbf{U} \begin{pmatrix} d^+_p(\mathbf{K}, s) \\ d^+_n(\mathbf{K}, s) \end{pmatrix} \mathbf{U}^+ = \mathbf{u} \begin{pmatrix} d^+_p(\mathbf{K}, s) \\ d^+_n(\mathbf{K}, s) \end{pmatrix} \]

Note \( \mathbf{u} \in \mathbf{U}_2 \) has 4 arbitrary continuous parameters, since \( \mathbf{u} \) is a complex 2x2 matrix \( \Rightarrow \) 8 real parameters but \( \mathbf{u}^+ \mathbf{u} = 1 \Rightarrow 4 \) constraints and therefore \( \mathbf{u} \) is defined in terms of 4 parameters.

Define \( \det \mathbf{u} = e^{-2i\theta} \), then

\[ \mathbf{u} = e^{-i\theta} \mathbf{U}_1 \] such that

\[ \det \mathbf{U}_1 = 1. \mathbf{U}_1 \text{ is an element of the group } SU_2 \text{ where} \]
\[ u = e^{-i\theta} \left( \begin{array}{cc}
\alpha^* & -\beta^* \\
\beta^* & \alpha^*
\end{array} \right) \quad 1\alpha^2 + 1\beta^2 = 1 \] (63)

\[ SU_2 \equiv \{ U | 2 \times 2 \text{ unitary matrices} \}
\]

\[ \text{det} U = 1 \}

\[ U_1 \text{ are given in terms of 3 arbitrary parameters.} \]

In general \( U_1 = e^{i \frac{2}{\hbar} \Theta} \)

\( \Theta \) are Pauli matrices,

\( \Theta_i \); \((i=1, 2, 3)\) are 3 real parameters.

\[ U_2 = SU_2 \otimes U_1 \]

Consider \( \Theta = 0 \Rightarrow U = e^{-i\theta} \)

\[ U(\theta, \Theta = 0) \left( \begin{array}{c}
d^+_p \\
d^+_n
\end{array} \right) U^+(\theta) = e^{-i\theta} \left( \begin{array}{c}
d^+_p \\
d^+_n
\end{array} \right) \]

Let \( U(\theta) = e^{i\theta B} \) and \( B = B^* \)

is an operator in Fock space.
Calculate

\[ -i \frac{d}{d\theta} \left[ U(\theta) \left( \frac{d^+_p}{d^+_n} \right) U^+(\theta) \right]_{\theta=0} \]

\[ = - \left( \begin{array}{c} d^+_p \\ d^+_n \end{array} \right) \]

\[ \Rightarrow \left[ B, \left( \begin{array}{c} d^+_p \\ d^+_n \end{array} \right) \right] = - \left( \begin{array}{c} d^+_p \\ d^+_n \end{array} \right) \]

Similarly

\[ \left[ B, \left( \begin{array}{c} b^+_p \\ b^+_n \end{array} \right) \right] = - \left( \begin{array}{c} b^+_p \\ b^+_n \end{array} \right) \]

or

\[ \left[ B, \left( \begin{array}{c} b^+_p \\ b^+_n \end{array} \right) \right] = \left( \begin{array}{c} b^+_p \\ b^+_n \end{array} \right) \]

We now know how B acts on proton and neutron states, since

\[ |p> = b^+_p |0> \quad , \quad |n> = b^+_n |0> \]

\[ \Rightarrow B|p> = |p> \quad , \quad B|n> = |n> \]
and \( |\bar{p}\rangle = d^+_p |0\rangle \), \( |\bar{n}\rangle = d^+_n |0\rangle \),

\[ \Rightarrow B |\bar{p}\rangle = -|\bar{p}\rangle, \quad B |\bar{n}\rangle = -|\bar{n}\rangle \]

and \( B |0\rangle = 0 \) was assumed.

Thus, \( B \) is the generator of Baryon number.

Now consider the \( SU_2 \) transformations.

\[ SU_2 \bigg/ U_1(\bar{\delta}) \psi_N(x) U_1^+(\bar{\delta}) = U_1 \psi_N(x) \]

\[ U(\theta, \bar{\delta}) = e^{i B \theta} U_1(\bar{\delta}) \]

and \( \det U_1 = 1 \).

These take protons into neutrons (this is isospin). Heisenberg assumed that the strong interactions (i.e., nuclear forces) were isospin invariant.
i.e. \[ U_I \ H_{st} \ U_I^+ = H_{st} \]

Note, we have discussed the action of \( U \) on free (or interaction picture) states. However, if we write \( H_{st} = H_0 + H_I \) and \( U_I \ H_0 \ U_I^+ = H_0 \), \( U_I \ H_I \ U_I^+ = H_I \) then it is easy to see that the Heisenberg states transform in the same way as the interaction picture states. If the interaction picture state is given by

\[ |\psi_t\rangle \equiv b_p^+ |10\rangle \]

then the fully interacting state is given by

\[ |\psi\rangle = e^{i H_{st} t} e^{-i H_0 t} |\psi_t\rangle \]
\[ U_I |p\rangle = U_I e^{i H_s t} e^{-i H_0 t} |p\rangle_0 \]

\[ \overset{t \to -\infty}{=} e^{i H_s t} e^{-i H_0 t} U_I |p\rangle_0 \]

since \[ [U_I, H_s] = \sum U_I, H_0 = 0 \]

and we evaluated \( U_I |p\rangle_0 \).

Also \[ [U_I, S] = 0 \]

\( \uparrow \) \( S \) operator

For homework you found that \( U_I(\vec{\theta}) = e^{-i \vec{I} \cdot \vec{\theta}} \)

with \[ [\vec{I}_i, \psi_N(x)] = -\frac{2i}{\hbar} \psi_N(x) \]

satisfies

\[ U_I(\vec{\theta}) \psi_N(x) U_I^{\dagger}(\vec{\theta}) \equiv U_I \psi_N(x) \]

with \( U_I \equiv e^{i \frac{\vec{I} \cdot \vec{\theta}}{\hbar}} \).
$I_i$ are the generators of isospin rotations, satisfying the Lie algebra

$$[I_i, I_j] = i \epsilon_{ijk} I_k.$$

$$I_i = \int d^3x \, \psi^+_N(x) \frac{\tau_i}{2} \psi_N(x)$$

and

$$j_i^\mu(x) = \bar{\psi}_N(x) y^\mu \frac{\tau_i}{2} \psi_N(x)$$

satisfies $\Theta_{\mu j_i^\mu}(x) = 0$ if isospin is conserved.

In terms of creation and annihilation operators

$$I_i = \int \frac{d^3k}{(2\pi)^3} \frac{\tau_i}{2} \sum_{s} \left( b^+_k(R,s) \frac{\tau_i}{2} \epsilon_m b_k(R,s) - \bar{d}^+_k(R,s) \frac{\tau_i}{2} \epsilon_m d_k(R,s) \right)$$

$$+ \left( \frac{\tau_i}{2} \right)_{ml}$$
Also (see pg. 62)

\[ U_I(\bar{\Theta}) \left( \begin{array}{c} 1p \\ 1n \end{array} \right) = U_I(\Theta) \left( \begin{array}{c} 1p \\ 1n \end{array} \right) \]

\[ U_I(\bar{\Theta}) \left( \begin{array}{c} 1p \\ 1n \end{array} \right) = \left( \begin{array}{c} 1p \\ 1n \end{array} \right) u_I^+(\bar{\Theta}) \]

\[ \text{Note } [U_I(\bar{\Theta}), H_{st}] = 0 \]

\[ \Rightarrow [I_i, H_{st}] = 0 \]

i.e. Differentiate \( \Theta_i = \hat{\Theta}_i \alpha \)

in \( U_I(\bar{\Theta}) H_{st} U_I^+(\bar{\Theta}) = H_{st} \)

by \( \alpha \) and let \( \alpha \to 0 \).

with \( \hat{\Theta}_i \) arbitrary.