Feynman rules (in momentum space)

1. Internal photon line
   \[ -\frac{i\gamma \mu}{p^2 + i\epsilon} \quad \mu \rightarrow \mu \]

2. Internal fermion line
   \[ i \left( \frac{p + m}{p^2 - m^2 + i\epsilon} \right) \alpha \beta \]

3. Vertex
   \[ -ie\gamma^\mu \alpha \beta \]

4. (2\pi)^4 \delta^4 (\sum_{\text{out}} p^\mu - \sum_{\text{in}} p_i^\mu)

Draw all possible connected diagrams. Assign momenta to each internal line such that momentum is conserved at every vertex.
5) Integrate over momenta for every closed loop
\[ \int \frac{d^4p}{(2\pi)^4} \]
(i.e. these momenta are not constrained by momentum conservation.)

6) \((-1)^L \times \text{# closed fermion loops.}\)

7) \(\frac{1}{n!} \times \text{order of perturbation theory}\)
combatoric factor
for each graph with same topology
(i.e. counts the number of independent Wick contractions giving the same graph)
\(\pm 1\) for relative sign of Wick contraction.

8) Wave functions for incoming and outgoing lines.
Example

Bhabha scattering

\[ e^+ e^- \rightarrow e^+ e^- \]

Momenta

\[ P_1, P_2, P_3, P_4 \]

Helicity

\[ S_1, S_2, S_3, S_4 \]

There are two Feynman diagrams

1. \[ P_1 \rightarrow X_1 \rightarrow Z_1 \rightarrow Z_2 \rightarrow X_2 \rightarrow P_2 \]

2. \[ P_1 \rightarrow X_1 \rightarrow Z_1 \rightarrow Z_2 \rightarrow X_2 \rightarrow P_2 \]

S-channel diagram

\[ S = (P_1 + P_2)^2 \]

\[ t = (P_3 - P_1)^2 \]
These graphs are derived from the 4 pt. Green's function

\[ (\psi_0, T(\Psi(x_2) \Psi(x_1) \Psi(x_4) \Psi(x_3)) \psi_0) \]

\[ \Rightarrow \frac{1}{2!} (\Phi_0, T(\Psi(x_2) \Psi(x_1) \Psi(x_4) \Psi(x_3)) \times \Psi(z_1)^\mu \Psi(z_1)^\nu \Psi(z_2)^\mu \Psi(z_2)^\nu \times A_\mu(z_1) A_\nu(z_2) \Phi_0) \]

where in the 2nd term the fields are implicitly interaction picture operators.

The two inequivalent contractions responsible for graphs 1+2 are given below.

\[ A_\mu(z_1) A_\nu(z_2) \times \]

\[ \begin{align*}
\text{1:} & \quad \Psi(x_2) \Psi(x_1) \Psi(x_4) \Psi(x_3) \Psi(z_1)^\mu \Psi(z_1)^\nu \Psi(z_2)^\mu \Psi(z_2)^\nu \\
\text{2:} & \quad \text{(diagram shown)}
\end{align*} \]

\[ \begin{align*}
\text{2:} & \quad \text{(diagram shown)} \\
\text{2:} & \quad \text{combinatorial factors}
\end{align*} \]
For each graph there is a combinatoric factor of 2 (interchange 2\(_1\leftrightarrow 2\_2\)) and a relative sign difference.

In momentum space we find

\[
T = (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \times
\left\{ \begin{array}{c}
\left[ + \bar{u}(p_4, s_4) \gamma^\mu u(p_2, s_2) \bar{v}(p_1, s_1) \gamma^\nu v(p_3, s_3) \right]
\end{array} \right.
\]

\[\times \left[ -i g_{\mu\nu} \frac{1}{(p_1 - p_3)^2 + ie} \right] ( + ie)^2 \]

\[= \left[ - \bar{v}(p_1, s_1) \gamma^\mu u(p_2, s_2) \bar{u}(p_4, s_4) \gamma^\nu v(p_3, s_3) \right]
\]

\[\times \left[ -i g_{\mu\nu} \frac{1}{(p_1 + p_2)^2 + ie} \right] ( + ie)^2 \]

\[= T\left( -, p_4, s_4; +, p_3, s_3; -, p_2, s_2; +, p_1, s_1 \right)\]
These are the two Feynman diagrams for the process \( A \) \( e^+e^- \rightarrow e^+e^- \). A crossed process is given by the following Feynman diagrams for \( e^-e^- \rightarrow e^-e^- \) \( B \).
It is obtained diagrammatically by reversing the incoming and outgoing anti-particle lines.

It can be shown order by order in perturbation theory that this crossed diagram $\boxed{B}$ can be obtained from the original diagrams $\boxed{A}$ by continuing the momenta $p_1 \to -p_3'$ and $p_3 \to -p_1'$. 
I now want to make this property of relativistic field theory plausible. I will show that to this order the amplitudes for \( A \)

\[
T(-, p_4, s_4; +, p_3, s_3; -, p_2, s_2; +, p_1, s_1)
\]

becomes upon crossing

\[
T(-, p_4, s_4; +, -p_1', s_3; -, p_2, s_2; +, -p_3', s_1)
\]

\[
\equiv T(-, p_4, s_4; -, p_3', -s_3; -, p_2, s_2; -, p_1', -s_1)
\]

where \( s_3' = -s_3 \) and \( s_1' = -s_3 \)

and the last amplitude is given by the Feynman diagrams for the crossed process \( B \).

**Proof:** First consider the photon propagators in \( A \). They are given by

\[
\frac{1}{s + i\epsilon} \quad \text{and} \quad \frac{1}{t + i\epsilon}
\]

respectively.
Upon crossing

\[ S = (p_1 + p_2)^2 \rightarrow (p_2 - p_3)^2 \equiv (p_1' - p_4)^2 \equiv u \]

i.e. \( S \rightarrow u \)

for \( A \)

for \( B \)

\[ t = (p_1 - p_3)^2 \rightarrow (p_1' - p_3')^2 \equiv t \]

i.e. \( t \rightarrow t \)

for \( A \)

for \( B \)

Thus the propagators of diagram \( A \) go into the propagators of diagram \( B \).

Now consider the spinors.

Show that

\[ u(E_1, \overrightarrow{p}_1, s_1) \rightarrow u(-E_3', \overrightarrow{p}_3', s_1) \]

\[ \equiv \left( \frac{1}{2s_i N(p_3, s_1)} \right) V(E_3', \overrightarrow{p}_3', s_1) \]

where \( \left( \frac{1}{2s_i N(p_3, s_1)} \right) \) is a phase factor.

Thus from our Feynman rules, the wave function for an incoming particle with momentum \( p_1 \) and spin \( s_1 \) becomes, upon crossing, the wave
function for an outgoing anti-particle with momentum $p_3'$ and spin $-s_1$. Graphically

\[ s = -\frac{1}{2} \quad p \rightarrow -p \quad \begin{cases} \text{in state} \end{cases} \]

\[ s = +\frac{1}{2} \quad \begin{cases} \text{out state} \end{cases} \]

\[ \text{under crossing} \]

proof:

\[ \begin{cases} (p - m) \ u(p', s) = 0 \\ (p + m) \ v(p', s) = 0 \end{cases} \]

\[ \Rightarrow \quad p u_{\mu} \rightarrow -p u_{\mu} \]

\[ u(E, p', s) \rightarrow u(-E, -p', s) = v(E, p', s) \]

i.e. satisfy same equation

also

\[ \frac{p \cdot \rho}{2} \begin{cases} \{ u(p', s) \} \end{cases} = s \begin{cases} \{ v(-p', s) \} \end{cases} \]
Therefore
\[- \frac{\mathbf{p} \cdot \mathbf{p}}{2} \left\{ \begin{array}{l} u(-E,-\mathbf{p},s) \\
\mathcal{V}(E,\mathbf{p},s) \end{array} \right\} = s \left\{ \begin{array}{l} u(-E,-\mathbf{p},s) \\
\mathcal{V}(E,\mathbf{p},s) \end{array} \right\}\]
where taking \( E \to -E \) in the first equation doesn't do anything.

Thus \( \mathcal{V}(\mathbf{p},s) \) is an eigenstate of helicity with eigenvalue \(-s\) and
\[u(-E,-\mathbf{p},s) = N \mathcal{V}(E,\mathbf{p},s)\]
where \( N \) is a normalization factor.

Consider
\[u(\mathbf{p},s) = \sqrt{E+m} \left( \begin{array}{c} \chi_{\mathbf{p}}(s) \\
\frac{\mathbf{p} \cdot \mathbf{p}}{E+m} \chi_{\mathbf{p}}(s) \end{array} \right)\]
with \( \mathbf{p} \cdot \mathbf{p} \chi_{\mathbf{p}}(s) = 2s \mathbf{p} \cdot \mathbf{p} \chi_{\mathbf{p}}(s) \).

Then
\[u(-E,-\mathbf{p},s) = \sqrt{m-E} \left( \begin{array}{c} \chi_{-\mathbf{p}}(s) \\
\frac{\mathbf{p} \cdot \mathbf{p}}{E-m} \chi_{-\mathbf{p}}(s) \end{array} \right)\]
but \( \chi_{-\mathbf{p}}(s) = \chi_{\mathbf{p}}(-s) \) and
\[ u(-E, -\vec{p}, s) = \frac{\sqrt{m^2 - E^2}}{\sqrt{E + m}} \left( \begin{array}{c} \chi^\rho(-s) \\ \frac{\vec{\sigma} \cdot \vec{p}}{E^2 - m^2} \chi^\rho(-s)(E + m) \end{array} \right) \]

and

\[ E^2 - m^2 = \vec{p}^2 \implies \sqrt{m^2 - E^2} = i / \vec{p}^1 \]

\[ \implies u(-E, -\vec{p}, s) = i / \vec{p}^1 \left( \begin{array}{c} \chi^\rho(-s) \\ \frac{\vec{\sigma} \cdot \vec{p}}{i \vec{p}^1} \chi^\rho(-s)(E + m) \end{array} \right) \]

\[ = i \left( \begin{array}{c} -\frac{\vec{\sigma} \cdot \vec{p}}{2s \sqrt{E + m}} \chi^\rho(-s) \\ 2s \sqrt{E + m} \chi^\rho(-s) \end{array} \right) \]

\[ = \frac{1}{2si} \sqrt{E + m} \left( \begin{array}{c} \frac{\vec{\sigma} \cdot \vec{p}}{E + m} \chi^\rho(-s) \\ \chi^\rho(-s) \end{array} \right) \]

using

\[ N(\hat{p}, s) \chi^\rho(-s) = \sigma_2 \chi^{\rho^*}(s) \]

and

\[ V(\vec{p}, s) = \sqrt{E + m} \left( \begin{array}{c} \frac{\vec{\sigma} \cdot \vec{p}}{E + m} \sigma_2 \chi^{\rho^*}(s) \\ \sigma_2 \chi^{\rho^*}(s) \end{array} \right) \]

\[ \implies u(-E, -\vec{p}, s) = N \cdot V(E, \vec{p}, s) \]

and

\[ N = \left( \frac{1}{2si} \right) N(\hat{p}, s) \]
Magnetic moments in QED

\[ p, s \rightarrow p', s' \]

\[ q = p' - p \]

\[ H_I = e A_\mu \bar{\psi} \gamma^\mu \psi \]

Consider the matrix element of \( H_I \) [point particle]

\[ (p', s') \langle p', s' | H_I | q, s ; p, s \rangle \]

\[ = e \epsilon \epsilon' (q^\alpha, s) \bar{u}(p', s') \gamma^\mu \Omega u(p, s) \]

**useful identity**

\[ 2m \gamma^\mu \frac{\not{p}'}{k'} \frac{\not{p}}{k} = \frac{p' \gamma^\mu + \gamma^\mu p}{m^2} \]

on shell

\[ = \frac{1}{2} \left\{ \gamma^\mu, \gamma^\mu \right\} + \frac{1}{2} \left[ \gamma^\mu, \not{p} \right] + \frac{1}{2} \left[ \gamma^\mu, \gamma^\mu \right] - \frac{1}{2} \left[ \gamma^\mu, \gamma^\mu \right] \]

\[ = (p' + p)^\mu - \frac{1}{2} \left[ \gamma^\mu, \gamma^\mu \right] q^\nu \]
\[
\sum_{\mu} = \frac{i}{4} \sum_{\mu} g_{\mu} \gamma^\mu
\]

\[\Rightarrow 2m \gamma^\mu = (p' + p)^\mu + 2i \sum_{\mu} g_{\mu} \gamma^\mu\]

\[\gamma^\mu = \frac{(p' + p)^\mu + i \sum_{\mu} g_{\mu} \gamma^\mu}{2m}\]

In the non-relativistic limit:

\[p'^\mu = p^\mu = mg_0\]
\[p'^\mu \gamma^\mu = mu^\mu\]

\[\Rightarrow 1^{st\ term\ in\ (A)}\ is\ charge\ interaction\]
\[2^{nd\ term\ is\ magnetic\ moment\ interaction}\]

**Proof:**

Consider 2nd term \( [\Omega = -1] \)

\[\Rightarrow (p', s', 1 \mathbb{R}^4; g, a; p, s)\]
\[-i e \frac{\epsilon_{\mu}(g, a)}{m} \overline{u}(p', s') \gamma_{\mu} u(p, s)\]
but
\[ e^{i q \cdot x} \delta_\mu (q, \lambda) Z_\nu = - \partial_\nu A_\mu (x) \]

Thus \((p', s') \mid H \frac{1}{2} \left( g, \lambda ; p, s \right) \mid p, s \rangle \)

\[ \frac{e}{m} \left( p', s' \mid \partial_\nu A_\mu \right. \left. \overline{\psi} \Sigma^\mu \psi \left| g, \lambda; p, s \right. \right) \]

\[ = \frac{-e}{2m} \left( p', s' \mid F_{\mu \nu} \overline{\psi} \Sigma^{\mu \nu} \psi \left| g, \lambda; p, s \right. \right) \]

but \( F_{ij} \equiv \epsilon_{ijk} B_k \)

\[ \Sigma_{ij}^g = \frac{1}{2} \epsilon_{ijk} \Sigma_k \]

\[ \Rightarrow \quad F_{ij} \Sigma_{ij}^g = \epsilon_{ijk} \epsilon_{ijl} \frac{1}{2} B_k \Sigma_l \]

\[ = \frac{-e}{2m} B \cdot \Sigma \]

\[ \Rightarrow \quad \left( p', s' \mid H \frac{1}{2} \left( g, \lambda; p, s \right) \right) \]

\[ \left( p', s' \mid \frac{-e}{2m} \overline{\psi} \cdot B \cdot \Sigma \psi \left| g, \lambda; p, s \right. \right) \]

Recall spin operator \( S = \frac{\Sigma}{2} \)
\[ \Rightarrow (p', s' - \frac{ge}{2m} \vec{B} \cdot \vec{4S} 4 I_{1/2}, J; \tau; p, s) \]

\[ \approx (p', s' | N \tau I_{1/2}, J; \tau; p, s) \]

or

\[ \mathcal{H}_I = -\frac{ge}{2m} \vec{B} \cdot \vec{4S} 4 \]

where \[ g = 2 \]

This is the so-called Dirac magnetic moment of point-like particle (electron, muon, ...)

\[ \vec{m} = \frac{ge \hbar \vec{S}}{2m} \quad g = 2 \]

Bohr magneton

\[ M_B = \frac{e \hbar}{2m_e} \approx 5.8 \times 10^{-11} \text{ MeV/T} \]

Nuclear magneton

\[ M_N = \frac{e \hbar}{2m_p} \approx 3.2 \times 10^{-14} \text{ MeV/T} \]