Basics of Complex Numbers (I)

1. General

- $i \equiv \sqrt{-1}$, so $i^2 = -1$, $i^3 = -i$, $i^4 = 1$ and then it starts over again.
- Any complex number $z$ can be written as the sum of a real part and an imaginary part:
  $$z = \text{Re } z + i \text{Im } z,$$
  where the numbers or variables in the $[]$’s are real. So $z = x + yi$ with $x$ and $y$ real is in this form but $w = 1/(a + bi)$ is not (see ’Rationalizing’ below). Thus, $\text{Im } z = y$, but $\text{Re } w \neq 1/a$.
- **Complex Conjugate:** The complex conjugate of $z$, which is written as $z^*$, is found by changing the sign of every $i$ in $z$:
  $$z^* = \text{Re } z - i \text{Im } z,$$
  so if $z = 1/(a + bi)$, then $z^* = 1/(a^* - bi)$.
  Note: There may be “hidden” $i$’s in the variables; if $a$ is a complex number, then $z^* = 1/(a^* - b i)$.
- **Magnitude:** The magnitude squared of a complex number $z$ is:
  $$zz^* \equiv |z|^2 = |\text{Re } z|^2 - (i)^2 |\text{Im } z|^2 = |\text{Re } z|^2 + |\text{Im } z|^2 \geq 0,$$
  where the last equality shows that the magnitude is positive (except when $z = 0$).
  **Basic rule:** if you need to make something real, multiply by its complex conjugate.

2. Rationalizing: We can apply this rule to “rationalize” a complex number such as $z = 1/(a + bi)$. Make the denominator real by multiplying by the complex conjugate on top and bottom:
  $$\frac{1}{a + bi} \cdot \frac{a - bi}{a - bi} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} + i \frac{-b}{a^2 + b^2},$$
  so $\text{Re } z = a/(a^2 + b^2)$ and $\text{Im } z = -b/(a^2 + b^2)$.

3. The Complex x–y Plane

- **Rectangular form:** Any complex number $z$ can be uniquely represented as a point in the $x$–$y$ plane, where the $x$–coordinate is $\text{Re } z$ and the $y$–coordinate is $\text{Im } z$ (see figure).
  - You can think of $i$ as a unit vector in the “imaginary” ($y$) direction.
  - The magnitude of $z$ is just the length of the vector from the origin.
- **Polar form:** We can also write $z$ in polar form as:
  $$z = r e^{i\theta} = r \cos \theta + i r \sin \theta,$$
  where $r$ and $\theta$ are real and equal to the length and angle of the vector.
– The complex conjugate of \( z = re^{i\theta} \) is \( z^* = re^{-i\theta} \).
– Thus the magnitude is \( |z| = \sqrt{zz^*} = r \).
– Rationalizing:
\[
\frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}.
\]

4. Multiplying Complex Numbers

• Multiplication is distributive: \((a + bi) \times (c + di) = (ac - bd) + i(ad + bc)\).
• In polar form, we multiply the \( r \)'s and add the \( \theta \)'s: if \( z_1 = r_1e^{i\theta_1} \) and \( z_2 = r_2e^{i\theta_2} \), then \( z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)} \).

5. Euler’s Theorem and other Goodies:

• From the polar form, we have:
\[
e^{i\theta} = \cos \theta + i \sin \theta .
\]
• Special values:
\[
e^{2\pi i} = 1 \quad e^{i\pi} = -1 \quad e^{i\pi/2} = i \quad e^{i\pi/4} = 1/\sqrt{2} + i/\sqrt{2}
\]
• We can rewrite \( \sin \) and \( \cos \):
\[
\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i},
\]
which can be very useful, since it is generally easier to work with exponentials then trigonometric functions.
• DeMoivre’s Theorem:
\[
z^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n[\cos(n\theta) + i \sin(n\theta)] .
\]

We can also write the theorem in the form:
\[
z^{1/n} = r^{1/n}[\cos(\theta/n) + i \sin(\theta/n)] ,
\]
which is great for taking the square root, cube root, etc. of complex numbers!