Physics 5300, Theoretical Mechanics  Spring 2015

Assignment 5 solutions

The problems numbers below are from Classical Mechanics, John R. Taylor, University Science Books (2005).

Problem 1  Taylor 10.10

Solution: (a)

\[ I = \frac{M}{L} \int_{x=0}^{L} dx x^2 = \frac{ML^3}{L} \cdot \frac{3}{3} = \frac{ML^2}{3} \quad (1) \]

(b)

\[ I = 2 \frac{M}{L} \int_{x=0}^{\frac{L}{2}} x^2 dx = \frac{2ML^3}{L24} = \frac{ML^2}{12} \quad (2) \]

Problem 2  Taylor 10.12

Solution: We first compute \( I \) for rotation about an axis parallel to the \( z \) axis, passing through one corner. We will then use the parallel axis theorem to get \( I \) around an axis passing through the centroid.

Let the length of the prism be \( L \). The area of each side is \( 2a \). The height is

\[ h = \sqrt{4a^2 - a^2} = \sqrt{3}a \quad (3) \]

The area is

\[ A = \frac{1}{2}(2a)\sqrt{3}a = \sqrt{3}a^2 \quad (4) \]

The volume is then

\[ V = \sqrt{3}a^2L \quad (5) \]

and the density is

\[ \rho = \frac{M}{V} = \frac{M}{\sqrt{3}a^2L} \quad (6) \]

Let the \( y \) axis be along the perpendicular of the triangle, with \( y = 0 \) being the vertex of the triangle. The range of \( y \) is

\[ 0 \leq y \leq \sqrt{3}a \quad (7) \]
At any given value of \( y \), the range of \( x \) is
\[
-\frac{1}{\sqrt{3}} y \leq x \leq \frac{1}{\sqrt{3}} y
\]  
(8)

The distance squared from the \( z \) axis is \( x^2 + y^2 \). Thus
\[
I = 2\rho \int_{z=0}^{L} dz \int_{y=0}^{\sqrt{3}a} dy \int_{x=0}^{\frac{1}{\sqrt{3}} y} dx (x^2 + y^2)
\]  
(9)

The first part is
\[
I_1 = 2\rho \int_{z=0}^{L} dz \int_{y=0}^{\sqrt{3}a} dy \int_{x=0}^{\frac{1}{\sqrt{3}} y} dx \cdot x^2 = 2\rho L \int_{y=0}^{\sqrt{3}a} dy \frac{x^3}{3} \bigg|_{x=0}^{\frac{1}{\sqrt{3}} y}
\]  
(10)
\[
= 2\rho L \int_{y=0}^{\sqrt{3}a} dy \frac{y^3}{9\sqrt{3}} = 2\rho L \frac{1}{9\sqrt{3}} \frac{y^4}{4} \bigg|_{y=0}^{\sqrt{3}a} = 2\rho L \frac{1}{9\sqrt{3}} \frac{9a^4}{4} = \frac{1}{2\sqrt{3}} \rho La^4 = \frac{1}{2\sqrt{3}} \frac{M}{3a^2L} La^4 = \frac{1}{6} Ma^2
\]  
(11)

The second part is
\[
I_2 = 2\rho \int_{z=0}^{L} dz \int_{y=0}^{\sqrt{3}a} dy y^2 \int_{x=0}^{\frac{1}{\sqrt{3}} y} dx = 2\rho \int_{z=0}^{L} dz \int_{y=0}^{\sqrt{3}a} dy y^2 x \bigg|_{x=0}^{\frac{1}{\sqrt{3}} y}
\]  
(12)
\[
= 2\rho \int_{z=0}^{L} dz \int_{y=0}^{\sqrt{3}a} dy y^3 \frac{1}{\sqrt{3}} = 2\rho L y^4 \frac{1}{\sqrt{3}} \bigg|_{y=0}^{\sqrt{3}a} = 2\rho L \frac{9a^4}{4} = \frac{3\sqrt{3}}{2} \rho La^4 = \frac{3\sqrt{3}}{2} \frac{M}{\sqrt{3}a^2L} La^4 = \frac{3}{2} Ma^2
\]  
(13)

The centroid is at a distance
\[
2 \sqrt{3} a = \frac{2}{\sqrt{3}} a
\]  
(14)

from the vertex. Thus we need to subtract an amount
\[
M \frac{4}{3} a^2
\]  
(15)

from the sum of the above two terms. This gives
\[
I = \frac{3}{2} Ma^2 + \frac{1}{6} Ma^2 - \frac{4}{3} Ma^2 = \frac{1}{3} Ma^2
\]  
(16)

The products of inertia will vanish by symmetry, if we compute them around the midpoints of the prism.

Problem 3  Taylor 10.15
Solution: (a) Let the axis of rotation be the $z$ axis, and the cube extend from 0 to $a$ on each axis. The density is

$$\rho = \frac{M}{a^3}$$  

(17)

We get

$$I = \rho \int_{z=0}^{a} dz \int_{y=0}^{a} dy \int_{x=0}^{a} dx (x^2 + y^2)$$  

(18)

We have

$$\rho \int_{z=0}^{a} dz \int_{y=0}^{a} dy \int_{x=0}^{a} dx x^2 = \rho \int_{z=0}^{a} dz \int_{y=0}^{a} dy \frac{a^3}{3} = \rho \frac{a^5}{3} = \frac{Ma^2}{3}$$  

(19)

The other integral gives the same result, so we get

$$I = \frac{2Ma^2}{3}$$  

(20)

(b) The height of the center of mass is $\frac{a}{\sqrt{2}}$ above the table. At the end, the center of mass is at a height $\frac{a}{2}$. Thus the PE lost is

$$Mga\left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right) = Mga\frac{\sqrt{2} - 1}{2}$$  

(21)

The KE gained is

$$\frac{1}{2}I\omega^2$$  

(22)

Thus we get

$$\frac{1}{2} \frac{2Ma^2}{3} \omega^2 = Mga\frac{\sqrt{2} - 1}{2}$$  

(23)

$$\omega^2 = \frac{g}{a^2} \frac{3}{2} (\sqrt{2} - 1)$$  

(24)

Problem 4 Taylor 10.21

Solution: If we have a diagonal term like $i = x, j = x$ then we get

$$I_{xx} = \int \rho (x^2 + y^2 + z^2 - x^2) = \int \rho (y^2 + z^2)$$  

(25)

which is correct. If we have an off diagonal term, then we get

$$I_{xy} = \int \rho (-xy)$$  

(26)
which is correct as well.

Problem 5  Taylor 10.35

Solution:  We have

\[ I_{xx} = \sum_i m_i (y^2 + z^2) = 2m[a^2 + a^2] + 3m[a^2 + a^2] = 10ma^2 \]  \hspace{1cm} (27)

\[ I_{yy} = \sum_i m_i (x^2 + z^2) = m[a^2] + 2m[a^2] + 3m[a^2] = 6ma^2 \]  \hspace{1cm} (28)

\[ I_{zz} = \sum_i m_i (x^2 + y^2) = m[a^2] + 2m[a^2] + 3m[a^2] = 6ma^2 \]  \hspace{1cm} (29)

\[ I_{xy} = -\sum_i m_i xy = m[0] = 0 \]  \hspace{1cm} (30)

\[ I_{yz} = -\sum_i m_i yz = -m[0] - 2m[a^2] - 3m[-a^2] = ma^2 \]  \hspace{1cm} (31)

\[ I_{xz} = -\sum_i m_i xz = -m[0] - 2m[0] = 0 \]  \hspace{1cm} (32)

Thus

\[ I = ma^2 \begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & 1 \\ 0 & 1 & 6 \end{pmatrix} \]  \hspace{1cm} (33)

The eigenvalues are found from the equation (keeping apart a factor of \( ma^2 \))

\[ \text{Det} \begin{pmatrix} 10 - \lambda & 0 & 0 \\ 0 & 6 - \lambda & 1 \\ 0 & 1 & 6 - \lambda \end{pmatrix} = 0 \]  \hspace{1cm} (34)

\[ (10 - \lambda)[(6 - \lambda)^2 - 1] = 0 \]  \hspace{1cm} (35)

One solution is

\[ \lambda = 10 \]  \hspace{1cm} (36)

The other solutions are given by

\[ (6 - \lambda)^2 = 1, \hspace{0.5cm} 6 - \lambda = \pm 1, \hspace{0.5cm} \lambda = 5, \hspace{0.5cm} \lambda = 7 \]  \hspace{1cm} (37)

Thus the moments of inertia are

\[ I_1 = 10ma^2, \hspace{0.5cm} I_2 = 5ma^2, \hspace{0.5cm} I_3 = 7ma^2 \]  \hspace{1cm} (38)
The eigenvectors are given by
\[
\begin{pmatrix}
10 & 0 & 0 \\
0 & 6 & 1 \\
0 & 1 & 6 \\
\end{pmatrix}
\begin{pmatrix}
V_x \\
V_y \\
V_z \\
\end{pmatrix}
= 10
\begin{pmatrix}
V_x \\
V_y \\
V_z \\
\end{pmatrix}
\] (39)

Let \( V_z = 1 \). Then we get
\[
6V_y + V_z = 10V_y, \quad -4V_y + V_z = 0, \quad V_y = \frac{V_z}{4}
\] (40)

We also get
\[
V_y + 6V_z = 10V_z, \quad V_y = -4V_z
\] (41)

Thus we get the eigenvector
\[
(1, 0, 0)
\] (42)

For \( \lambda = 5 \) we get
\[
\begin{pmatrix}
10 & 0 & 0 \\
0 & 6 & 1 \\
0 & 1 & 6 \\
\end{pmatrix}
\begin{pmatrix}
V_x \\
V_y \\
V_z \\
\end{pmatrix}
= 5
\begin{pmatrix}
V_x \\
V_y \\
V_z \\
\end{pmatrix}
\] (43)

Thus
\[
10V_x = 5V_x, \quad V_x = 0
\] (44)
\[
6V_y + V_z = 5V_y, \quad V_y = -V_z
\] (45)
\[
V_y + 6V_z = 5V_z, \quad V_y = -V_z
\] (46)

Thus we can take the eigenvector
\[
(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})
\] (47)

For \( \lambda = 7 \) we get
\[
\begin{pmatrix}
10 & 0 & 0 \\
0 & 6 & 1 \\
0 & 1 & 6 \\
\end{pmatrix}
\begin{pmatrix}
V_x \\
V_y \\
V_z \\
\end{pmatrix}
= 7
\begin{pmatrix}
V_x \\
V_y \\
V_z \\
\end{pmatrix}
\] (48)

Thus
\[
10V_x = 7V_x, \quad V_x = 0
\] (49)
\[
6V_y + V_z = 7V_y, \quad V_y = V_z
\] (50)
\[
V_y + 6V_z = 7V_z, \quad V_y = V_z
\] (51)

Thus we can take the eigenvector
\[
(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})
\] (52)
Problem 6  Taylor 10.40

Solution: (a) We have
\[ I_1 \dot{\omega}_1 - (I_2 - I_3) \omega_2 \omega_3 = 0 \] (53)
Multiplying by \( L_1 = I_1 \omega_1 \) gives
\[ L_1 I_1 \dot{\omega}_1 - I_1 (I_2 - I_3) \omega_2 \omega_3 \omega_1 = 0 \] (54)
We have two similar equations for the other components. Adding them, the second terms in these equations are seen to add to zero. The first terms add to
\[ L_1 (I_1 \dot{\omega}_1) + L_2 (I_2 \dot{\omega}_2) + L_3 (I_3 \dot{\omega}_3) = 0 \] (55)
This is
\[ I_1 \dot{I}_1 + I_2 \dot{I}_2 + I_3 \dot{I}_3 = 0 \] (56)
\[ \frac{d}{dt} \left[ \frac{1}{2} (I_1^2 \dot{I}_2 + I_3^2 \dot{I}_3) \right] = 0 \] (57)
which is
\[ \frac{d}{dt} L^2 = 0 \] (58)
so the magnitude of the angular momentum remains unchanged.

(b) We have
\[ I_1 \dot{\omega}_1 - (I_2 - I_3) \omega_2 \omega_3 = 0 \] (59)
Multiplying by \( \omega_1 \) gives
\[ I_1 \dot{\omega}_1 \omega_1 - (I_2 - I_3) \omega_2 \omega_3 \omega_1 = 0 \] (60)
When we add the three equations, the second terms cancel. The first terms give
\[ I_1 \frac{1}{2} \frac{d}{dt} (\omega_1^2) + I_2 \frac{1}{2} \frac{d}{dt} (\omega_2^2) + I_3 \frac{1}{2} \frac{d}{dt} (\omega_3^2) = \frac{d}{dt} T = 0 \] (61)