Capacitor at high frequencies *a la* Feynman

Consider a parallel-plate capacitor with cylindrical plates of radius $a$ and a distance $d$ apart. For simplicity we will assume that the capacitor does not contain any dielectric material. Recall that the capacitance is given by $\epsilon_0 A/d$ where $A = \pi a^2$; at finite frequencies we argued that the impedance is $1/(i\omega C)$. Let us study the the fields inside a capacitor when it is driven by an AC source with frequency $\omega$. As the voltage alternates the charges on the plates oscillate and change sign. This means that the electric field inside is varying as a function of time; recall that the “displacement current,” $\epsilon_0 \partial \vec{E}/\partial t$, introduced by James Clerk Maxwell implies that a changing electric field induces a a magnetic field; the changing magnetic field thus created induces an electric field by Faraday’s Law and so on! We will calculate these fields in a “simple-minded way.”

We start out with a uniform electric field along the $z$-direction (perpendicular to the plates) given by

$$\vec{E} = \hat{z} E_0 e^{i\omega t}$$

where we will use complex notation and for variety let us choose the actual electric field to be given by the real part of the expression above. We start from

$$\nabla \times \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$$

where we have used $\vec{J} = 0$ and $\mu_0 \epsilon_0 = 1/c^2$. Applying Stokes theorem we write the integral form (you should remember both forms!)

$$\oint_C \vec{B} \cdot d\vec{\ell} = \frac{1}{c^2} \int_S \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a}.$$

We will choose the the loop $C$ to be a circle with radius $r$ and centered on the axis with is plane parallel to the plates. The electric field and $d\vec{a}$ point along $\hat{z}$ and by symmetry we expect the magnetic field to be tangential to the circle\(^1\) and the line integral can be easily evaluated. For large plates and $r << a$ we expect $\vec{E}$ to be uniform and we find

$$B = \frac{2\pi r}{2\pi r} = \pi r^2 i\omega E_0 e^{i\omega t}.$$  

\(^1\)Note that the displacement current $\partial \epsilon_0 \vec{E}/\partial t$, along $\hat{z}$, acts like an ordinary current and produces the field as in the case of Ampère’s Law.
This means that we find a harmonic magnetic field:

$$\vec{B} = \hat{\phi} \frac{i\omega r}{2c^2} E_0 e^{i\omega t}. $$

Note that the strength depends linearly on $\omega$ and $r$.

The changing magnetic field produces an electric field and we can compute this using Faraday’s Law. So the calculation we did in assuming that the electric field is independent of $r$ is incorrect and we have to calculate the corrections. So we write

$$E = E_0 e^{i\omega t} + E_2$$

where $E_2$ is the correction due to the magnetic field.\(^2\) We choose $E_2$ so that it vanishes at $r = 0$; if there is a non-zero contribution we can include it in $E_0$. Now we start from Faraday’s Law

$$\oint_C \vec{E} \cdot d\vec{a} = -\frac{\partial}{\partial t} \int_S \vec{B} \cdot d\vec{a}.$$

Now we choose the Amperian loop carefully remembering (i) the direction of the electric field (ii) symmetry and (iii) the requirements that there is a net flux through the loop and the integrals are easy to evaluate. Since the electric field $E_2$ points along $z$, vanishes at $r = 0$ we choose $C$ to consist of a line along the $z$-axis at $r = 0$, a radial path outward at the upper plate, a vertical path down at the radius $r$ from the upper to the lower plate and a radial path back to the axis at the lower plate. From the figure, the infinitesimal area is $dz \, dr$ the flux is given by

$$\int_0^h dz \int_0^r dr \frac{i\omega r}{2c^2} E_0 e^{i\omega t} = \frac{i\omega h r^2}{4c^2} E_0 e^{i\omega t}.$$ 

The line integral of $E_2$ is $-E_2 h$ (explain this clearly!) Therefore we have

$$E_2(r) h = \frac{\partial}{\partial t} \left[ \frac{i\omega h r^2}{4c^2} E_0 e^{i\omega t} \right]$$

and we have

$$E_2(r) = -\frac{\omega^2 r^2}{4c^2} E_0 e^{i\omega t}$$

which means that the electric field at a distance $r$ from the axis is given by

$$\vec{E} = \hat{z} E_0 e^{i\omega t} \left( 1 - \frac{\omega^2 r^2}{4c^2} \right).$$

\(^2\)I have labelled it $E_2$ to keep to Feynman’s notation as far as possible; he calls the first term $E_1$.\)
So the electric field is not uniform across the capacitor and has a parabolic form with the maximum at the axis. What does all this mean from the point of view of circuit theory? When there is a current through this circuit element there is an associated magnetic field and one can compute the coefficient of self-inductance \( \Phi_B = LI \). So in the presence of the alternating emf the capacitance behaves as an inductance which becomes increasingly important as the frequency increases.

One can continue with this, i.e., use the correction to the electric field to calculate the correction to \( \vec{B} \) and repeat it \textit{ad infinitum}. Feynman actually does the next step; the answer if one continues is

\[
E = E_0 e^{i\omega t} \left[ 1 - \frac{1}{1!} \left( \frac{\omega r}{2c} \right)^2 + \frac{1}{2!} \left( \frac{\omega r}{2c} \right)^4 - \frac{1}{3!} \left( \frac{\omega r}{2c} \right)^6 + \cdots \right].
\]

The infinite series is actually \( J_0(\omega r/c) \) which is the Bessel function of zeroth order. One can solve the partial differential equation by separation of variables and obtain this result but I find the method used by Feynman instructive.

**Aside:** At leading order the charge on the capacitor plate is given by \( \epsilon_0 (\pi a^2) E_0 e^{i\omega t} \) from which one can calculate the current and equating \( LI \) to \( \Phi_B \) through the wedge-shaped surface from the expression calculated earlier we have

\[
L i\omega \epsilon_0 \pi a^2 E_0 e^{i\omega t} = \frac{i\omega \hbar a^2}{4c^2} E_0 e^{i\omega t} \equiv \Phi_B
\]

which implies

\[
L = \frac{\mu_0}{4\pi} h.
\]

The impedance due to this inductance is small compared to the impedance \( 1/(i\omega C) \) if

\[
\omega \frac{\mu_0}{4\pi} h \ll \frac{\hbar}{\omega \epsilon_0 \pi a^2} \Rightarrow \frac{a}{c} \ll \frac{1}{\omega}
\]

the time it takes for light to travel across a characteristic distance is small compared to \( 1/\omega \). At very high frequencies this condition will not be satisfied and the response of the capacitor is inductive.

We will assume that the solution for the electric is be given by

\[
\vec{E} = \hat{z} E_0 e^{i\omega t} J_0 \left( \frac{\omega r}{c} \right).
\]
The first zero of $J_0$ occurs when the argument is 2.405. Note that for $a = 10\text{cm}$ the frequency should be $2.3 \times 10^9\text{s}^{-1}$ for the zero to occur at half the radius. So this means that at high enough frequencies the electric field at the center points one way and it points in the opposite direction at the edge!

Now the electric field vanishes at some value of $r$ where $\omega r/c = 2.405$; there is no field on a circle coaxial with the plates for all values of $z$. Now we can put thin metal sheet shaped like a can without the caps and fit it just inside the capacitor. No currents will flow through it because the electric fields vanish at that value of $r$. Now we can remove the outside of the capacitor plates and throw away the connections. The electric and magnetic fields will oscillate! How does this occur?

Changing $\vec{E}$ makes a $\vec{B}$ and the changing $\vec{B}$ makes $\vec{E}$.

This self-sustained oscillations occurs only for $r = 2.405\omega/c$. If the radius is fixed the can is resonant only at a particular frequency. Comment on the eigenvalue problem and boundary conditions, etc.

Also when we remove the outside of the can and the associated electric and magnetic fields currents actually flow in the sides of the electric can. This means that there can be dissipation and so the resonance has a width.
Consider a transmission with inductance per unit length of $L_0$ and a capacitance per unit length of $C_0$. Let the current in the hot conductor be $I(x)$ and the voltage along the line be $V(x)$. If the current in the line is varying as a function of time the inductance will yield a voltage drop across a small segment of the line from $z$ to $z + dz$. We have

$$V(z + dz) - V(z) = -(L_0 dz) \frac{\partial I}{\partial t}.$$  

The changing current yields a gradient in the voltage.

Since the transmission line has a capacitance per unit length the changing voltage as a function of distance implies a change in the charge stored in the region; (remember $Q = CV$). The change in the charge in a region means a gradient in the current: charge conservation or the continuity equation, The net outflow of current should equal the rate of change in the charge in the region of length $dz$:

$$I(z + dz) - I(z) = -(C_0 dz) \frac{\partial V}{\partial t}.$$  

These two equations lead immediately to

\begin{align*}
\frac{\partial V}{\partial z} &= -L_0 \frac{\partial I}{\partial t} \quad (1) \\
\frac{\partial I}{\partial z} &= -C_0 \frac{\partial V}{\partial t}. \quad (2)
\end{align*}

Together they yield the wave equation for both $I(z,t)$ and $V(z,t)$.

Note that the solutions are of the form

$$V = V_0 e^{ikz-i\omega t} \quad \text{and} \quad I = I_0 e^{ikz-i\omega t}.$$  

Substituting this into the first of the pair of first-order partial differential equations we have

$$ik V_0 = i\omega L_0 I_0$$

and using $\omega = ck$ with $c = 1/\sqrt{L_0 C_0}$ we obtain

$$V_0 = \sqrt{\frac{L_0}{C_0}} I_0.$$  

Thus the impedance of the transmission line $Z_0$ is $\sqrt{L_0/C_0}$.  


We have \( V_0^+ = Z_0 I_0^+ \) where the superscript indicates that the wave is travelling to the right. It is worth noting that for wave travelling to the left we have

\[
V = V_0^- e^{-ikz-i\omega t} \quad \text{and} \quad I = I_0^- e^{-ikz-i\omega t}.
\]

This yields, upon using \( \partial V/\partial z = -L_0 \partial I/\partial t \),

\[
-ik V_0 e^{-ikz-i\omega t} = L_0 i\omega e^{-ikz-i\omega t} \Rightarrow V_0^- = -Z_0 I_0^-.
\]

Recall that for the coaxial cable we had

\[
L_0 = \frac{\mu_0}{2\pi} \log(b/a) \quad \text{and} \quad C_0 = \frac{2\pi \epsilon_0}{\log(b/a)}.
\]

Therefore, the impedance is given by

\[
Z = \sqrt{\frac{L_0}{C_0}} = \frac{1}{2\pi} \sqrt{\frac{\mu_0}{\epsilon_0}} \log(b/a) = (60 \, \Omega) \log(b/a).
\]

Recall that \( 1/(4\pi \epsilon_0) = 8.9918 \times 10^9 \). The dimensions of \( \epsilon_0 \) are \( \text{Farad/m} \); that of \( 1/\epsilon_0 \) are \( \Omega \text{m/s} \). Since \( \log(b/a) \) is of the order of \( 2-3 \) the typical impedance of a coaxial cable is around 100\( \Omega \).
Consider an LCR circuit with an initial charge $Q(t = 0) = Q_0$ on the capacitor and no initial current: $I(t = 0) = 0$. In the absence of any sources the equation for the circuit is

$$L \frac{dI}{dt} + IR + \frac{Q}{C} = 0$$

which upon differentiation becomes (using $I = \dot{Q}$ where the dot denotes differentiation)

$$L\ddot{I} + R\dot{I} + \frac{I}{C} = 0.$$ 

This is the one canonical second-order ODE with constant coefficients you should be able to solve with your eyes closed. Since the sum of the function $I(t)$, its derivative and second derivative should add up to zero for all times a reasonable guess for the solution is $I(t) = A e^{\alpha t}$ where $\alpha$ is yet to be determined. Substituting into the equation we find a quadratic equation (a lot easier than an ODE):

$$L\alpha^2 + R\alpha + \frac{1}{C} = 0.$$ 

The two solutions are

$$\alpha_\pm = \frac{1}{2L} \left(-R \pm \sqrt{R^2 - \frac{4L}{C}}\right).$$

First convince yourself that both roots are negative. Sometimes it is convenient to divide the ODE by $L$, define $\omega_0^2 \equiv 1/(LC)$ and a damping constant $\Gamma = R/L$ and write it as

$$\ddot{I} + \Gamma \dot{I} + \omega_0^2 I = 0.$$ 

There are three cases:

*overdamped* when the argument of the square root is positive, i.e., $R^2 > 4L/C$, the resistance is large. In this case the solution is

$$I(t) = A_+ e^{\alpha_+ t} + A_- e^{\alpha_- t}$$

which we can write in the following form:

$$I(t) = A_+ e^{-t/\tau_+} + A_- e^{-t/\tau_-}$$

where $\tau_\pm = 1/|\alpha_\pm|$ since the alphas are negative. The term that dominates is the one with the smaller $\alpha$ or larger time scale $\tau_-$ since it decays more slowly. Note that the
two unknown constants are determined by the initial conditions.

**critically damped** when \( R^2 = 4L/C \) and in this case there is a degeneracy and the decay time is

\[
\tau = \frac{1}{|\alpha|} = \frac{L}{2R}.
\]

If you are careful you will observe that we are missing a solution: second-order ODE has *two* solutions (theorem from mathematics.) You can (and should) verify that the other solution is \( t e^{-Rt/(2L)} \). Note that the decay time is shorter than in the overdamped case! So in the critically damped case the decay is the fastest – don’t trust the terminology! A typical example of a system close to critical damping is the system of shock absorbers in your automobile; if it is overdamped the response to bumps will be slow!

Finally there is the *underdamped* case when \( 4L/C > R^2 \). Writing the two roots as

\[
\alpha \pm \frac{R}{2L} \pm i\beta \quad \text{where} \quad \beta = \frac{1}{2} \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}
\]

the solution becomes

\[
I(t) = e^{-Rt/(2L)} \left( A_+ e^{i\beta t} + A_- e^{-i\beta t} \right).
\]

Of course in this case the coefficients can be complex but since \( I(t) \) has to be real we have \( A_+ = A_-^* \). Writing \( A_+ = Ae^{i\phi} \) the solution is

\[
I(t) = A e^{-Rt/(2L)} \cos(\beta t + \phi).
\]