Spin-1/2 dynamics

The intrinsic angular momentum of a spin-1/2 particle such as an electron, proton, or neutron assumes values $\pm \hbar/2$ along any axis. The spin state of an electron (suppressing the spatial wave function) can be described by an abstract vector or ket a concrete realization of which is a two-component column vector.

The intrinsic angular momentum of a particle is a vector operator whose components obey the standard angular momentum commutation relations. Since a spin-1/2 particle has two possible results of a measurement they can be described by $2 \times 2$ matrices. Recall the Pauli representation:

$$\vec{S} = \frac{\hbar}{2} \vec{\sigma}$$

(1)

where $\vec{\sigma}$ are the Pauli spin matrices defined by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(2)

The choice makes $S_z$ diagonal. Recall the logic of how these are determined. We can write down $S_z$ since it is diagonal and the diagonal elements are the eigenvalues. The eigenvectors are given by

$$|z+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |z-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$  

We use the definitions of $S_+$ and $S_-:

$$S_+ |z+\rangle = 0 \quad \text{and} \quad S_+ |z-\rangle = |z+\rangle$$

$$S_- |z+\rangle = |z-\rangle \quad \text{and} \quad S_- |z-\rangle = 0$$

allow us to find that

$$S_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad S_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$  

(Knowing $S_+$ we can find $S_-$ since it is the Hermitian conjugate of $S_+$. Since $S_\pm = S_x \pm iS_y$, we can find $S_x$ and $S_y$.

Some simple properties that you should verify and learn to use:

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I.$$  

$$\sigma_x \sigma_y = i\sigma_z, \quad \sigma_y \sigma_z = i\sigma_x, \quad \sigma_z \sigma_x = i\sigma_y.$$  

So the "standard" basis corresponds to spin up and down along the z-axis. In particular if the particle is in a state described by the ket

$$|s\rangle = \rightarrow \begin{pmatrix} a \\ b \end{pmatrix}$$

(3)

then the probability of finding $+\hbar/2$ upon making a measurement of the spin along the z-axis is simply $a^*a = |a|^2$. Absolutely explicitly this probability is given by the squared absolute value of the "overlap" matrix element

$$\langle z+|s\rangle = (1, 0) \begin{pmatrix} a \\ b \end{pmatrix} = a.$$  

(4)
We study the effect of a magnetic field along the z-axis on a spin oriented along the x-axis initially. The solution to the more general problem follows the same logic but is algebraically more tedious. Quantum mechanically we start with the Hamiltonian for a magnetic field along the z-axis

\[ H = -\gamma B_0 S_z = -\omega_L S_z = -\frac{\hbar \omega_L}{2} \sigma_z = \begin{pmatrix} -\frac{\hbar \omega_L}{2} & 0 \\ 0 & \frac{\hbar \omega_L}{2} \end{pmatrix}. \]

Since we are given that the spin points up along \( \hat{x} \) initially we have

\[ |\chi(t = 0)\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}. \]

Recall that \( |\chi\rangle \) is the spinor wave function that contains all the information about the system (or more precisely an ensemble of identically prepared systems). We wish to find the state at time \( t \) given by \( |\chi(t)\rangle \).

### Find the eigenvalues and eigenvectors of the Hamiltonian:

We know that the eigenvalues of \( H \) are \( \pm \frac{\hbar \omega_L}{2} \) since it is diagonal. The corresponding eigenvectors are

\[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \] and \[ \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]

### Expand the initial state in terms of the eigenfunctions of \( H \):

We have, by inspection, \(^1\)

\[ |\chi(t - 0)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]

### Use the prescription for determining the state at time \( t \) by appending a factor of \( e^{-iEt/\hbar} \) appropriately:

Therefore, at time \( t \) we have

\[ |\chi(t)\rangle = e^{i\omega_L t/2} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{-i\omega_L t/2} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]

We find

\[ |\chi(t)\rangle = \begin{pmatrix} e^{i\omega_L t/2} \\ e^{-i\omega_L t/2} \end{pmatrix}. \]

Given \( |\chi(t)\rangle \) we can calculate expectation value of operators and the probability of making a measurement and finding a specific value. For example, suppose we wish to find the probability of measuring \( S_x \) and finding the value \( +\hbar/2 \). As always we find the eigenvector corresponding to the eigenvalue \( +\hbar/2 \) denoted by \( |x+\rangle \). Then the probability of measuring \( +\hbar/2 \) along \( x \) at time \( t \) is given by how much the state at that time \( |\chi(t)\rangle \) "looks like" the eigenvector \( |x+\rangle \). This is given by the overlap \( \langle x+ |\chi(t)\rangle \) and the probability by \( |\langle x+ |\chi(t)\rangle|^2 \). We can also determine expectation values in a straightforward manner as illustrated below.

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\(^1\)Please be clear about what the procedure is when it is not evident from inspection.
The expectation value of the spin along $x$ given by $\langle \chi(t)|S_x|\chi(t)\rangle$ can be calculated. Note that if we measure the spin along the $x$-axis at time $t$ on an ensemble of identically prepared (at time $t = 0$) systems that evolve according to the given Hamiltonian the average value of the measurements is given by the expectation value.

Recall that $\langle \chi(t) \rangle$ is obtained by the Hermitian conjugate operation: transpose and complex conjugate.

Therefore, we have

$$\langle \chi(t)|S_x|\chi(t)\rangle = \frac{\hbar}{2} \left( \begin{array}{cc} e^{-i\omega_L t/2} & e^{i\omega_L t/2} \\ \sqrt{2} & \sqrt{2} \end{array} \right) \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \left( \begin{array}{c} e^{i\omega_L t/2} \\ \sqrt{2} \\ e^{-i\omega_L t/2} \sqrt{2} \end{array} \right)$$

$$= \frac{\hbar}{2} \left( \begin{array}{c} e^{-i\omega_L t/2} \\ \sqrt{2} \\ e^{i\omega_L t/2} \sqrt{2} \end{array} \right) \left( \begin{array}{c} e^{-i\omega_L t} \\ 1 \end{array} \right) \left( \begin{array}{c} e^{i\omega_L t} \sqrt{2} \\ e^{-i\omega_L t} \sqrt{2} \end{array} \right)$$

$$= \frac{\hbar}{2} \left( e^{-i\omega_L t} + e^{i\omega_L t} \right)$$

$$= \frac{\hbar}{2} \cos(\omega_L t).$$

(5)

Note that $\langle S_x \rangle(t)$ precesses in the plane perpendicular to the field. This corresponds to the classical result as we see below. Of course there is no real classical analog of spin-1/2 but we use the associated magnetic moment to investigate the effect.

**Classical physics:** Consider the dynamical problem of a spin-1/2 particle in a magnetic field. We will study the simplest case by choosing $\hat{z}$ along the magnetic field. We study the problem classically first. In a uniform magnetic field the magnetic moment $\vec{\mu}$ experiences a torque $\vec{\mu} \times \vec{B}$. Since the magnetic moment is proportional to the angular momentum we have $\vec{\mu} = \gamma \vec{J}$. Recall that the rate of change of the angular momentum is the torque:

$$\frac{d\vec{J}}{dt} = \vec{\mu} \times \vec{B} = \gamma \vec{J} \times \vec{B}.$$  

(7)

Choosing $\vec{B} = B_0 \hat{z}$ and defining $\omega_L = \gamma B_0$ we can write down the equations for each component:

$$\dot{J}_x = \omega_L J_y, \quad \dot{J}_y = -\omega_L J_x, \quad \text{and} \quad \dot{J}_z = 0.$$  

Clearly $J_z$ the projection along the magnetic field is a constant in time. We solve the other two equations by a useful trick. Multiplying the equation for $J_y$ by $i$ and adding to the $J_x$ equation we have

$$\dot{J}_x + iJ_y = \omega_L (J_y - iJ_x) = -i\omega_L (J_x + iJ_y).$$

Recall that $\dot{f} = -i\omega_L f$ is easily solved as $f(t) = f(0) e^{-i\omega_L t}$. Check that this obeys the equation and the initial condition at $t = 0$. Thus we obtain

$$J_x(t) + iJ_y(t) = (J_x(0) + iJ_y(0)) e^{-i\omega_L t}.$$
Let us choose \( J_y(0) = 0 \) so that the moment is oriented in the \( xz \)-plane initially. We have
\[
J_x(t) + iJ_y(t) = J_x(0) e^{-i\omega_L t} \Rightarrow J_x(t) = J_x(0) \cos(\omega_L t) \text{ and } J_y(t) = -J_x(0) \sin(\omega_L t).
\]
Thus we have the magnetic moment vector describing a cone with its tip moving in a circle with frequency \( \omega_L \). This is referred to as Larmor precession.

**Problem:** Consider a spin-1/2 particle with spin pointing up along \( \hat{n} \) given by
\[
\hat{n} = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta).
\]
What is the 2-component vector (called a spinor) that corresponds to this state?

It is important to note that we are measuring the spin along \( \hat{n} \) and the operator corresponding to this observable is \( \vec{S} \cdot \hat{n} \). It is given by
\[
\vec{S} \cdot \hat{n} = \frac{\hbar}{2} (\sin\theta \cos\phi \sigma_x + \sin\theta \sin\phi \sigma_y + \cos\theta \sigma_z)
\]
\[
\vec{S} \cdot \hat{n} = \frac{\hbar}{2} \left[ \begin{pmatrix} 0 & \sin\theta \cos\phi \\ \sin\theta \cos\phi & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i\sin\theta \sin\phi \\ i\sin\theta \sin\phi & 0 \end{pmatrix} + \begin{pmatrix} \cos\theta & 0 \\ 0 & -\cos\theta \end{pmatrix} \right]
\]
\[
\Rightarrow \vec{S} \cdot \hat{n} = \frac{\hbar}{2} \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}.
\]

We need to find the eigenvalues and eigenvectors. Consider the matrix without the factor of \( +\hbar/2 \). Note that the trace defined to be the sum of the diagonal matrix elements. The trace is also the sum of the eigenvalues; denoting them by \( \lambda_1 \) and \( \lambda_2 \) we have \( \lambda_1 + \lambda_2 = 0 \). The determinant is easily calculated to be \( -1 \) and this is the product of the eigenvalues. Thus we find \( \lambda_1 \lambda_2 = -1 \). Together we have \( \lambda_1 = 1 \) and \( \lambda_2 = -1 \). Thus the eigenvalues of \( \vec{S} \cdot \hat{n} \) are \( \pm \hbar/2 \). This shows the result that the spin measured along any arbitrary axis yields only two possible values, \( \pm \hbar/2 \). This is an amazing feature of quantum mechanics. Please spend a few minutes thinking about what happens classically.

Let us denote the eigenvectors by \( |\hat{n}+\rangle \) and \( |\hat{n}−\rangle \). We can determine these easily:
\[
|\hat{n}+\rangle = \begin{pmatrix} \cos\theta e^{-i\phi} \\ \sin\theta \end{pmatrix} \quad \text{and} \quad |\hat{n}−\rangle = \begin{pmatrix} \sin\theta e^{-i\phi} \\ -\cos\theta \end{pmatrix}
\]

\(^2\)The general case is easily solved by choosing
\[
J_x(0) + iJ_y(0) = J_\perp(0) e^{i\phi}
\]
to find
\[
J_x(t) = J_\perp(0) \cos(\omega_L t - \phi) \text{ and } J_y(t) = -J_\perp(0) \sin(\omega_L t - \phi).
\]

\(^3\)We have for example
\[
\frac{\hbar}{2} \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} a \\ b \end{pmatrix}.
\]
Thus we have (canceling \( \hbar/2 \))
\[
\cos\theta a + \sin\theta e^{-i\phi} b = a \Rightarrow \frac{a}{b} = \frac{\sin\theta}{1 - \cos\theta} e^{-i\phi} = \frac{2 \sin(\theta/2) \cos(\theta/2)}{2 \sin^2(\theta/2)} e^{-i\phi} = \frac{\cos(\theta/2)}{\sin(\theta/2)} e^{-i\phi}
\]
where we have used the half-angle formulae. We have chosen \( a = \cos(\theta/2) e^{-i\phi} \) and \( b = \sin(\theta/2) \).
This is the solution to Problem 4.31 in Griffiths (page 160) except for an overall phase factor of $\exp(i\phi)$. Feynman in Vol. III gives a more symmetrical formulae by multiplying by $e^{i\phi/2}$ (Equation 10.30):

\[
|\hat{n}+\rangle = \left( \frac{\cos \frac{\theta}{2} e^{-i\phi/2}}{\sin \frac{\theta}{2} e^{i\phi/2}} \right) \quad \text{and} \quad |\hat{n}-\rangle = \left( \frac{\sin \frac{\theta}{2} e^{-i\phi/2}}{-\cos \frac{\theta}{2} e^{i\phi/2}} \right)
\] (12)

**Problem:** Given a spin in the state $|z+\rangle$, i.e., pointing up along the $z$-axis what are the probabilities of measuring $\pm \hbar/2$ along $\hat{n}$?

The probability of measuring up is given by $|\langle \hat{n} | z+ \rangle|^2$. This is

\[
|\langle \hat{n} | z+ \rangle|^2 = |\cos(\theta/2)e^{i\phi}|^2 = \cos^2(\theta/2).
\]

The probability of measuring $-\hbar/2$ along $\hat{n}$ given that the spin points up along $z$ is $\sin^2(\theta/2)$. Please verify this explicitly.

How does one interpret this classically? Classically the angular momentum along $\hat{n}$ is $(\hbar/2) \cos \theta$. We have to compare the classical result with the quantum mechanical expectation value $\langle \hat{S} \cdot \hat{n} \rangle$. The expectation value or the mean values is given by the sum of the to possible values $\pm \hbar/2$ multiplied by their corresponding probabilities:

\[
\frac{\hbar}{2} \cos^2(\theta/2) + \left( -\frac{\hbar}{2} \right) \sin^2(\theta/2) = \frac{\hbar}{2} (\cos^2(\theta/2) - \sin^2(\theta/2)) = \frac{\hbar}{2} \cos \theta.
\]

This is an example of how expectation values conform to classical expectations in this the most quantum mechanical of systems.