Angular Momentum Eigenvalues

Define raising and lowering operators by

\[ J_\pm = J_1 \pm iJ_2 = J_x \pm iJ_y \]  \hspace{1cm} (1)

We note that \[ [J_z, J_\pm] = \pm \hbar J_\pm \] as can be easily verified:

\[ J_z (J_x \pm iJ_y) - (J_x \pm iJ_y) J_z = (J_z J_x - J_x J_z) \mp i(J_y J_z - J_z J_y) \]
\[ = i\hbar J_y \mp i(i\hbar J_x) = \pm \hbar (J_x \pm iJ_y) = \pm \hbar J_\pm \]  \hspace{1cm} (2)

Next define

\[ J^2 = J_x^2 + J_y^2 + J_z^2 \]  \hspace{1cm} (3)

from which it is easy to check that

\[ [J^2, J_z] = 0 \text{ and } [J^2, J_\pm] = 0 . \]  \hspace{1cm} (4)

Note also that

\[ J_+ J_- = J_x^2 + J_y^2 + \hbar J_z = J^2 - J_z^2 + \hbar J_z \text{ and } \]
\[ J_- J_+ = J_x^2 + J_y^2 - \hbar J_z = J^2 - J_z^2 - \hbar J_z \]  \hspace{1cm} (5)

Let \(|\alpha, m\rangle\) be a normalized eigenfunction of \(J^2\) with eigenvalue \(\alpha\) and of \(J_z\) with eigenvalue \(m\).

\[ J^2 |\alpha, m\rangle = \hbar^2 \alpha |\alpha, m\rangle \]
\[ J_z |\alpha, m\rangle = m\hbar |\alpha, m\rangle \]  \hspace{1cm} (6)

Claim 1: \(J_+ |\alpha, m\rangle\) is an eigenfunction of \(J_z\) with eigenvalue \((m + 1)\hbar\).

Proof:

\[ J_z J_+ |\alpha, m\rangle = (J_+ J_z + \hbar J_+) |\alpha, m\rangle = (m + 1)\hbar J_+ |\alpha, m\rangle \]

where we have used \([J_+, J_z] = \hbar J_+\).

Claim 2: \(J_- |\alpha, m\rangle\) is an eigenfunction of \(J_z\) with eigenvalue \((m - 1)\hbar\).

Proof:

\[ J_z J_- |\alpha, m\rangle = (J_- J_z - \hbar J_-) |\alpha, m\rangle = (m - 1)\hbar J_- |\alpha, m\rangle \]

where we have used \([J_-, J_z] = -\hbar J_-\).

Claim 3: \(\alpha \geq m^2\): Proof: We show this by noting that

\[ \langle \alpha, m | J^2 - J_z^2 |\alpha, m\rangle = \alpha - m^2 = \langle \alpha, m | (J_x^2 + J_y^2) |\alpha, m\rangle \geq 0 \]

where the last inequality follows since \(J_x^2 + J_y^2\) is a positive definite operator. This is assigned as a Homework problem;

Hint: Note that \(\langle \alpha, m | J_z^2 |\alpha, m\rangle\) is the squared magnitude of a ket.
The above shows that for a given \( \alpha \) there is a maximum value of \( m \) which we denote by \( m_{\text{max}} \). Therefore,
\[
J_+ |\alpha, m_{\text{max}}\rangle = 0 .
\]

Claim 4: \( \alpha = m_{\text{max}}(m_{\text{max}} + 1) \).

Proof:
\[
J_- J_+ |\alpha, m_{\text{max}}\rangle = 0
\]
\[
= (J^2 - J_z^2 - \hbar J_z) |\alpha, m_{\text{max}}\rangle = \hbar^2 (\alpha - m_{\text{max}}^2 - m_{\text{max}}) |\alpha, m_{\text{max}}\rangle
\]

Now \( J_- \) lowers the eigenvalue \( m \) by 1 and eventually it becomes negative and the absolute magnitude of the lowest value is again bounded by \( \alpha \geq m_{\text{min}}^2 \).
\[
J_- |\alpha, m_{\text{min}}\rangle = 0
\]

Note also that since \( J_- \) lowers the value of have
\[
m_{\text{min}} = m_{\text{max}} - k
\]
where \( k \) is an integer.

We also have Claim 5: \( \alpha = m_{\text{min}}(m_{\text{min}} - 1) \).

Proof:
\[
J_+ J_- |\alpha, m_{\text{min}}\rangle = 0
\]
\[
= (J^2 - J_z^2 + \hbar J_z) |\alpha, m_{\text{min}}\rangle = \hbar^2 (\alpha - m_{\text{min}}^2 + m_{\text{min}}) |\alpha, m_{\text{min}}\rangle
\]

Claim 6: We have \( m_{\text{max}} = -m_{\text{min}} \).

Proof: Since \( \alpha = m_{\text{min}}(m_{\text{min}} - 1) = m_{\text{max}}(m_{\text{max}} - 1) \) we obtain
\[
m_{\text{max}}^2 - m_{\text{min}}^2 = -m_{\text{max}} - m_{\text{min}}
\]
and therefore, \( m_{\text{max}} + m_{\text{min}} = 0 \); otherwise, \( m_{\text{max}} = m_{\text{min}} - 1 \), a contradiction.

Since we have \( m_{\text{max}} = -m_{\text{min}} \) and \( m_{\text{max}} - m_{\text{min}} = k \) where \( k \) is an integer we obtain \( m_{\text{max}} = k/2 \) i.e., \( m_{\text{max}} \) is either an integer or a half-integer!
The eigenfunctions of the angular momentum operator \( L_z \) is most easily calculated in spherical polar coordinates. Recall that \( L_z = x p_y - y p_x \). In spherical coordinates
\[
x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi \quad \text{and} \quad z = r \cos \theta.
\]
We also have since \( r^2 = x^2 + y^2 + z^2 \) the result \( \frac{\partial}{\partial x} = \frac{x}{r} \); given \( \tan \phi = y/x \) we obtain
\[
\frac{\partial \phi}{\partial x} = -\frac{y}{x^2 + y^2} \quad \text{and} \quad \frac{\partial \phi}{\partial y} = \frac{x}{x^2 + y^2};
\]
finally, \( \cos \theta = z/\sqrt{x^2 + y^2 + z^2} = z/r \) we get after some manipulations
\[
\frac{\partial \theta}{\partial x} = \frac{x z}{r^2 \sqrt{x^2 + y^2}} \quad \text{and} \quad \frac{\partial \theta}{\partial y} = \frac{y z}{r^2 \sqrt{x^2 + y^2}}.
\]
Thus
\[
\frac{\partial}{\partial y} = \frac{y}{r} \frac{\partial}{\partial r} + \frac{y z}{r^2 \sqrt{x^2 + y^2}} \frac{\partial}{\partial \theta} + \frac{x}{x^2 + y^2} \frac{\partial}{\partial \phi}
\]
and
\[
\frac{\partial}{\partial x} = \frac{x}{r} \frac{\partial}{\partial r} + \frac{x z}{r^2 \sqrt{x^2 + y^2}} \frac{\partial}{\partial \theta} - \frac{y}{x^2 + y^2} \frac{\partial}{\partial \phi}.
\]
Note that
\[-i\hbar \left[ x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right]\]
upon substitution and simplification leads to the compact result
\[
L_3 = -i\hbar \frac{\partial}{\partial \phi}
\]
which has the pleasing feature that \( \phi \) is the angle by which one rotates about the third \((z)\) axis and the angular momentum is conjugate to this angle. Now it is easy to obtain the eigenfunctions of this operator:
\[
L_z f(r, \theta, \phi) = \lambda f(r, \theta, \phi)
\]
which implies that
\[
f = g(r, \theta) e^{i\lambda \phi / \hbar}.
\]
Let us focus on the \( \phi \) dependence only. Since under \( \phi \to \phi + 2\pi \) we return the system back to the original position and we expect nothing to be changed we demand that \( \lambda \) is an integer which will be denoted by \( m \). This argument cannot be very convincing since \( \psi \) itself is not observable. It is better to use the fact that \( L_z \) is Hermitian - See next homework problem set. Thus the eigenfunctions of \( L_3 \) are \( \exp(i m \phi) \) for integral \( m \).