Some of these problems were considered for the second midterm

7.1) A precocious freshman comes to you and says, “I am confused by tunneling. Consider the rectangular potential barrier between \(-a\) and \(a\). When \(E < V_0\) we know that the wave function in the region \(-a < x < a\) is a linear combination of \(\exp(-\kappa x)\) and \(\exp(+\kappa x)\). Now I remember that the current vanishes when the wave function is real. I know that there is an incident current and there is a transmitted current because of tunneling. How can there be a current for \(x > a\) without a current in between? This quantum mechanics sure is weird.” Provide a lucid explanation for this thoughtful student. (4 points)

7.2) Revisiting an old friend, integration by parts: Verify that the momentum operator \(\hat{p}\) obeys
\[
\int_{-\infty}^{\infty} dx \left[ \hat{p} \psi(x) \right]^* \phi(x) = \int_{-\infty}^{\infty} dx \psi^* \left[ \hat{p} \phi(x) \right]
\]
for any normalizable \(\psi(x)\) and \(\phi(x)\). Technically what you have shown (not proved!) is that \(\hat{p}\) is a Hermitian operator. (2 points)

(7.3) This problem is relevant to the physics of the field-emission microscope. Consider a charged particle(electron) of charge \(-e\) and mass \(m\) in a potential

\[
V(x) = \begin{cases} 
0 & (x < 0) \\
V_0 - eE x & (x > 0) 
\end{cases}
\]

where we have denoted the electric field by \(E\) to avoid confusing it with the energy. Consider an energy \(0 < E < V_0\) so that the electron sees a triangular barrier. Find the classical turning points. Denote \(V_0 - E\) by \(W\). Find the probability for the electron to penetrate the barrier (defined by a crude version of the so-called WKB approximation) described in the lecture\(^1\):
\[
T \approx e^{-\frac{2}{\hbar} \int_{x_L}^{x_R} dx \sqrt{2m(V(x) - E)}}
\]
where \(x_L\) and \(x_R\) are the classical turning points. Evaluate the integral and find an expression for \(T\). Write \(T\) as
\[
T \approx e^{-\mathcal{E}_0/E}
\]
and for \(W = 2eV\) find a numerical value for \(\mathcal{E}_0\) in \(V/m\). How might one achieve such large fields? (8 points)

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\(^1\)Remind yourself of the justification of this result from the expression for the tunneling probability through a constant potential barrier described in the lectures and in the text. It is derived more carefully in Chapter 8, see Equations 8.2 and 8.22.
This was on a midterm exam at Harvard many years ago: Calculate \( < p >, < p^2 > \) and \( < p^3 > \) as function of time for the simple harmonic oscillator state given by

\[
\Psi(x, t) = \frac{1}{\sqrt{2}} \left( \psi_1(x)e^{-i\frac{3}{2}\omega t} + \psi_4(x)e^{-i\frac{9}{2}\omega t} \right).
\]

The notation is the same as in the text. Identify terms that vanish and only evaluate those that you need. (6 points)

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You are encouraged to think about this problem if you are going to graduate school in physics but please do not submit.

Consider a one-dimensional potential with \( V(x) = 0 \) for \( x < -a \) (denoted by region \( I \)) and for \( x > a \) (denoted by region \( III \)), i.e., outside the region \([-a, a]\). In the region in between (denoted by \( II \)) the potential is an arbitrary smooth function. For a plane wave \( e^{ikx} \) incident from the left let the wave function that solves the (time-independent) Schrödinger equation be

\[
\psi_I(x) = e^{ikx} + r_L e^{-ikx} \quad \text{and} \quad \psi_{III}(x) = t_L e^{ikx}.
\]

The form of the wave function in \( II \) is messy and unknown. For a plane wave \( e^{-ikx} \) incident from the right let the wave function in region I and III that solves the Schrödinger equation for the same energy be

\[
\psi_I(x) = t_R e^{-ikx} \quad \text{and} \quad \psi_{III}(x) = r_R e^{ikx} + e^{-ikx}.
\]

The goal of the problem is to relate \( t_R \) and \( r_R \) to \( t_L \) and \( r_L \) independently of the form of \( V \). The surprisingly general result can be derived by using two ingredients: (i) If \( \psi(x) \) is a solution to the Schrödinger equation so is \( \psi^*(x) \) assuming that the potential is real. (ii) Superposition principle: If \( \psi_1 \) and \( \psi_2 \) are solutions to the Schrödinger equation so is any linear combination with complex coefficients.