Lecture 3: R-L-C AC Circuits

AC (Alternative Current):
- Most of the time, we are interested in the voltage at a point in the circuit
  - will concentrate on voltages here rather than currents.
- We encounter AC circuits whenever a periodic voltage is applied to a circuit.
- The most common periodic voltage is in the form of a sine (or cosine) wave:
  \[ V(t) = V_0 \cos \omega t \quad \text{or} \quad V(t) = V_0 \sin \omega t \]

- \( V_0 \) is the amplitude:
  - \( V_0 = \text{Peak Voltage} \ (V_p) \)
  - \( V_0 = 1/2 \text{ Peak-to-Peak Voltage} \ (V_{PP}) \)
    - \( V_{PP} \): easiest to read off scope
  - \( V_0 = \sqrt{2} \ V_{RMS} = 1.41 \ V_{RMS} \)
    - \( V_{RMS} \): what multimeters usually read
\( \omega \) is the angular frequency:
- \( \omega = 2\pi f \), with \( f \) = frequency of the waveform.
- frequency (\( f \)) and period (\( T \)) are related by:
  \[ T \text{ (sec)} = \frac{1}{f \text{ (sec}^{-1})} \]
- Household line voltage is usually 110-120 \( V_{RMS} \) (156-170 \( V_p \)), \( f = 60 \text{ Hz} \).

- It is extremely important to be able to analyze circuits (systems) with sine or cosine inputs
- Almost any waveform can be constructed from a sum of sines and cosines.
- This is the “heart” of Fourier analysis (Simpson, Chapter 3).
- The response of a circuit to a complicated waveform (e.g. a square wave) can be understood by analyzing individual sine or cosine components that make up the complicated waveform.
- Usually only the first few components are important in determining the circuit’s response to the input waveform.

R-C Circuits and AC waveforms
- There are many different techniques for solving AC circuits
  - All are based on Kirchhoff’s laws.
  - When solving for voltage and/or current in an AC circuit we are really solving a differential eq.
  - Different circuit techniques are really just different ways of solving the same differential eq:
    - brute force solution to differential equation
    - complex numbers (algebra)
    - Laplace transforms (integrals)
We will solve the following RC circuit using the brute force method and complex numbers method.

Let the input (driving) voltage be \( V(t) = V_0 \cos \omega t \) and we want to find \( V_R(t) \) and \( V_C(t) \).

**Brute Force Method:** Start with Kirchhoff's loop law:
\[
V(t) = V_R(t) + V_C(t)
\]
\[
V_0 \cos \omega t = IR + Q/C
\]
\[
= RdQ(t)/dt + Q(t)/C
\]

- We have to solve an inhomogeneous D.E.
- The usual way to solve such a D.E. is to assume the solution has the same form as the input:
\[
Q(t) = \alpha \sin \omega t + \beta \cos \omega t
\]
- Plug our trial solution \( Q(t) \) back into the D.E.:
\[
V_0 \cos \omega t = \alpha R \omega \cos \omega t - \beta R \omega \sin \omega t + (\alpha/C) \sin \omega t + (\beta/C) \cos \omega t
\]
\[
= (\alpha R \omega + \beta/C) \cos \omega t + (\alpha/C - \beta R \omega) \sin \omega t
\]
\[
V_0 = \alpha R \omega + \beta/C
\]
\[
\alpha/C = \beta R \omega
\]
\[
\alpha = \frac{RC^2 \omega V_0}{1 + (RC \omega)^2}
\]
\[
\beta = \frac{CV_0}{1 + (RC \omega)^2}
\]
We can now write the solution for $V_C(t)$:

$$V_C(t) = \frac{Q}{C}$$

$$= \left(\alpha \sin \omega t + \beta \cos \omega t\right)/C$$

$$= \frac{RC\omega V_0}{1 + (RC\omega)^2} \sin \omega t + \frac{V_0}{1 + (RC\omega)^2} \cos \omega t$$

We would like to rewrite the above solution in such a way that only a cosine term appears.

- In this form we can compare it to the input voltage.

$$V_C(t) = \frac{V_0}{\sqrt{1 + (RC\omega)^2}} \left[ \frac{RC\omega}{\sqrt{1 + (RC\omega)^2}} \sin \omega t + \frac{1}{\sqrt{1 + (RC\omega)^2}} \cos \omega t \right]$$

- We get the above equation in terms of cosine only using the following basic trig:

  $$\cos(\theta_1 - \theta_2) = \sin \theta_1 \sin \theta_2 + \cos \theta_1 \cos \theta_2$$

- We can now define an angle such that:

  $$\cos \phi = \frac{1}{\sqrt{1 + (RC\omega)^2}}$$

  $$\sin \phi = \frac{RC\omega}{\sqrt{1 + (RC\omega)^2}}$$

  $$\tan \phi = RC\omega$$

  $$V_C(t) = \frac{V_0}{\sqrt{1 + (RC\omega)^2}} \cos(\omega t - \phi)$$

- $V_C(t)$ and $V_0(t)$ are out of phase.
Using the above expression for $V_C(t)$, we obtain:

$$V_R(t) = IR$$

$$= R \frac{dQ}{dt}$$

$$= RC \frac{dV_C}{dt}$$

$$= \frac{-RC \omega V_o}{\sqrt{1 + (RC \omega)^2}} \sin(\omega t - \phi)$$

We would like to have cosines instead of sines by using:

$$-\sin \theta = \cos(\theta + \frac{\pi}{2})$$

$$V_R(t) = \frac{RC \omega V_o}{\sqrt{1 + (RC \omega)^2}} \cos(\omega t - \phi + \frac{\pi}{2})$$

- $V_C(t)$, $V_R(t)$, and $I(t)$ are all out of phase with the applied voltage.
- $I(t)$ and $V_R(t)$ are in phase with each other.
- $V_C(t)$ and $V_R(t)$ are out of phase by $90^\circ$.
- The amplitude of $V_C(t)$ and $V_R(t)$ depend on $\omega$. 
Example: RC Circuit

![Diagram of RC Circuit]

- R1: 1E3 Ω
- C2: 1E-5 F
- Vp = 1 V
- Frequency: 60 Hz

**Graphs:**

1. \( V_{in} \) vs. Time in ms
2. \( V_{out} \) vs. Time in ms

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Solving circuits with complex numbers:

**PROS:**
- don't explicitly solve differential equations (lots of algebra).
- can find magnitude and phase of voltage separately.

**CONS:**
- have to use complex numbers!
- No “physics” in complex numbers.

What's a complex number? (see Simpson, Appendix E, P835)
- Start with $j = \sqrt{-1}$ (solution to $x^2 + 1 = 0$).
- A complex number can be written in two forms:
  - $X = A + jB$
    - $A$ and $B$ are real numbers
  - $X = R e^{i\phi}$
    - $R = (A^2 + B^2)^{1/2}$ and $\tan\phi = B/A$ (remember $e^{i\phi} = \cos\phi + j\sin\phi$)

Define the complex conjugate of $X$ as:
- $X^* = A - jB$ or $X^* = R e^{-j\phi}$

The magnitude of $X$ can be found from:
- $|X| = (XX^*)^{1/2} = (X^*X)^{1/2} = (A^2 + B^2)^{1/2}$

Suppose we have 2 complex numbers, $X$ and $Y$ with phases $\alpha$ and $\beta$ respectively,
- $Z = \frac{X}{Y} = \frac{|X|e^{j\alpha}}{|Y|e^{j\beta}} = \frac{|X|}{|Y|} e^{j(\alpha - \beta)}$
  - magnitude of $Z$: $|X|/|Y|$
  - phase of $Z$: $\alpha - \beta$

So why is this useful?
Consider the case of the capacitor and AC voltage:

\[ V(t) = V_0 \cos \omega t \]

\[ = \text{Re} \left( V_0 e^{j\omega t} \right) \]

\[ Q = CV \]

\[ I(t) = C \frac{dV}{dt} \]

\[ = -C \omega V_0 \sin \omega t \]

\[ = \text{Re} \left( j\omega CV_0 e^{j\omega t} \right) \]

\[ = \text{Re} \left( \frac{V_0 e^{j\omega t}}{1/j\omega C} \right) \]

\[ = \text{Re} \left( \frac{V}{X_C} \right) \]

- V and \( X_C \) are complex numbers

- We now have Ohm's law for capacitors using the capacitive reactance \( X_C \):

\[ X_C = \frac{1}{j\omega C} \]
We can make a similar case for the inductor:

\[ V = L \frac{dI}{dt} \]

\[ I(t) = \frac{1}{L} \int V \, dt \]

\[ = \frac{1}{L} \int V_0 \cos \omega t \, dt \]

\[ = \frac{V_0 \sin \omega t}{\omega L} \]

\[ = \text{Re} \left( \frac{V_0 e^{j\omega t}}{j\omega L} \right) \]

\[ = \text{Re} \left( \frac{V}{X_L} \right) \]

- V and \( X_L \) are complex numbers
- We now have Ohm’s law for inductors using the inductive reactance \( X_L \):
  \[ X_L = j\omega L \]
- \( X_C \) and \( X_L \) act like frequency dependent resistors.
  - They also have a phase associated with them due to their complex nature.
  - \( X_L \Rightarrow 0 \) as \( \omega \Rightarrow 0 \) (short circuit, DC)
  - \( X_L \Rightarrow \infty \) as \( \omega \Rightarrow \infty \) (open circuit)
  - \( X_C \Rightarrow 0 \) as \( \omega \Rightarrow \infty \) (short circuit)
  - \( X_C \Rightarrow \infty \) as \( \omega \Rightarrow 0 \) (open circuit, DC)
Back to the RC circuit.

- Allow voltages, currents, and charge to be complex:
  \[ V_{in} = V_0 \cos \omega t \]
  \[ = \text{Re} \left( V_0 e^{j\omega t} \right) \]
  \[ = \text{Re} \left( V_R + V_C \right) \]

- We can write an expression for the charge \((Q)\) taking into account the phase difference \((\phi)\) between applied voltage and the voltage across the capacitor \((V_C)\).
  \[ Q(t) = CV_C(t) \]
  \[ = A e^{j(\omega t - \phi)} \]
  - \(Q\) and \(V_C\) are complex
  - \(A\) and \(C\) are real

- We can find the complex current by differentiating the above:
  \[ I(t) = dQ(t)/dt \]
  \[ = j\omega A e^{j(\omega t - \phi)} \]
  \[ = j\omega Q(t) \]
  \[ = j\omega CV_C(t) \]

\[ V_{in} = V_C + V_R \]
\[ = V_C + IR \]
\[ = V_C + j\omega CV_C R \]
\[ V_C = \frac{V_{in}}{1 + j\omega RC} \]
\[ = \frac{V_{in}}{j\omega C} \left( \frac{1}{R + \frac{1}{j\omega C}} \right) \]
\[ = V_{in} \frac{X_C}{R + X_C} \]

- looks like a voltage divider equation!!!!!

- We can easily find the magnitude of \( V_C \):

\[ |V_C| = |V_{in}| \frac{|X_C|}{|R + X_C|} \]
\[ = \frac{V_0}{\omega C} \frac{1}{\sqrt{R^2 + (1/\omega C)^2}} \]
\[ = \frac{V_0}{\sqrt{1 + (RC\omega)^2}} \]

- same as the result on page 4.
Is this solution the same as what we had when we solved by brute force page 3?

\[ V_C = \text{Real} \left( \frac{V_{in}}{1 + j\omega RC} \right) \]

\[ = \text{Real} \left( \frac{V_0 e^{j\omega t}}{1 + j\omega RC} \right) \]

\[ = \text{Real} \left( \frac{V_0 e^{j\omega t}}{\sqrt{1 + (\omega RC)^2}} e^{j\phi} \right) \]

\[ = \text{Real} \left( \frac{V_0 e^{j(\omega t - \phi)}}{\sqrt{1 + (\omega RC)^2}} \right) \]

\[ = \frac{V_0 \cos(\omega t - \phi)}{\sqrt{1 + (\omega RC)^2}} \]

\[ \phi \text{ is given by } \tan\phi = \omega RC \]

\[ \text{YES the solutions are identical.} \]
We can now solve for the voltage across the resistor.

Start with the voltage divider equation in complex form:

\[ V_R = \frac{V_{in}R}{R + X_C} \]

\[ |V_R| = \frac{|V_{in}|R}{|R + X_C|} \]

\[ = \frac{V_0R}{\sqrt{R^2 + (1/\omega C)^2}} \]

\[ = \frac{V_0\omega RC}{\sqrt{1 + (\omega RC)^2}} \]

This amplitude is the same as the brute force differential equation case!

In adding complex voltages, we must take into account the phase difference between them.

the sum of the voltages at a given time satisfy:

\[ V_0^2 = |V_R|^2 + |V_C|^2 \]

\[ V_0 = |V_R| + |V_C| \]

**R-C Filters**

- Allow us to select (reject) wanted (unwanted) signals on the basis of their frequency structure.
- Allow us to change the phase of the voltage or current in a circuit.
- Define the gain \( G \) or transfer \( H \) function of a circuit:
  - \( G(j\omega) = H(j\omega) = V_{out}/V_{in} \) (\( j\omega \) is often denoted by \( s \)).
  - \( G \) is independent of time, but can depend on \( \omega, R, L, C \).
For an RC circuit we can define $G_R$ and $G_C$:

\[
G_R \equiv \frac{V_R}{V_{in}} = \frac{R}{R + X_C} = \frac{R}{R + 1/j\omega C}
\]

\[
G_C \equiv \frac{V_C}{V_{in}} = \frac{X_C}{R + X_C} = \frac{1/j\omega C}{R + 1/j\omega C}
\]

We can categorize the $G$'s as follows:

<table>
<thead>
<tr>
<th></th>
<th>$G_R$</th>
<th>$G_C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>High Frequencies</td>
<td>$\approx 1$, no phase shift</td>
<td>$\approx 1/j\omega CR \approx 0$, phase shift</td>
</tr>
<tr>
<td></td>
<td>High pass filter</td>
<td></td>
</tr>
<tr>
<td>Low Frequencies</td>
<td>$\approx j\omega CR \approx 0$, phase shift</td>
<td>$\approx 1$, no phase shift</td>
</tr>
<tr>
<td></td>
<td>Low pass filter</td>
<td></td>
</tr>
</tbody>
</table>

Decibels and Bode Plots:

- Decibel (dB) describes voltage or power gain:
  
  \[
  \text{dB} = 20 \log(V_{out}/V_{in})
  \]
  
  \[
  = 10 \log(P_{out}/P_{in})
  \]

- Bode Plot is a log-log plot with dB on the y axis and log($\omega$) or log($f$) on the x axis.
3 dB point or 3 dB frequency:
- also called break frequency, corner frequency, 1/2 power point
- At the 3 dB point:
  \[ \frac{V_{out}}{V_{in}} = \frac{1}{\sqrt{2}} \]  since \( 3 = 20 \log\left(\frac{V_{out}}{V_{in}}\right) \)
  \[ \frac{P_{out}}{P_{in}} = \frac{1}{2} \]  since \( 3 = 10 \log\left(\frac{P_{out}}{P_{in}}\right) \)

\( \omega RC = 1 \) for high or low pass filter
Phase vs frequency for capacitor

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