20. BEC with sodium. The second experiment to produce BEC in a dilute gas used sodium atoms. The number density was \( N/V = 10^{20} \text{ atoms/m}^3 \). The mass of a sodium atom is \( m = 3.82 \times 10^{-26} \text{ kg} \). As with the rubidium experiment, only one state of intrinsic angular momentum was populated.

(a) If the trap that confined the atoms were adequately approximated by a box with rigid walls, at what temperature would you expect BEC to set in (as one lowered the temperature)?

(b) How low a temperature would be required for 90 percent of the atoms to be in the single-particle ground state?

(c) The common, stable isotope of sodium has 12 neutrons and is the isotope referred to above: \(^{23}\text{Na}\). The unstable isotope \(^{21}\text{Na}\) has ten neutrons, the same nuclear spin, and a half-life of 23 seconds. In the following, suppress the possibility of radioactive decay.

A box of volume 1 cm\(^3\) contains \( 10^{14} \) sodium atoms at a temperature \( T = 1.3 \times 10^{-6} \text{ K} \). The atoms form a dilute gas, and only one state of intrinsic angular momentum is populated. Determine whether the heat capacity \( C_V \) is an increasing or decreasing function of temperature if

(i) all atoms are \(^{23}\text{Na}\) atoms;
(ii) half are \(^{23}\text{Na}\) and half are \(^{21}\text{Na}\).
Problem 20

(a) The critical temperature is

\[ kT_c = 5^{\frac{3}{2}} \left( \frac{\hbar^2}{2 \pi m} \right) \left( \frac{N}{V} \right)^{2/3} \]

Setting \( m = 3.82 \times 10^{-26} \) kg and \( n = 10^{20} / \text{m}^3 \), we obtain

\[ T_c = 1.75 \mu K \]

(b) The condensate fraction is

\[ \frac{N_0}{N} = \left( 1 - \frac{T}{T_c} \right)^{3/2} \]

This is equal to 0.90 at the temperature

\[ T = 0.068T_c = 0.12 \mu K \]

(c) The heat capacity \( C_v \) is an increasing function of \( T \) if \( T < T_c \) and a decreasing function of \( T \) if \( T > T_c \).

(i) \( C_v \) is an increasing function of \( T \) since 1.3 \( \mu K \) is less than \( T_c = 1.75 \mu K \)
(ii) The two isotopes are like 2 spin states. The critical temperature for 2 spin states is

\[ kT_c = \left( 2^8 \right)^{2/3} \frac{h^2}{2\pi m} \left( \frac{N}{V} \right)^{2/3} \]

\[ T_c = 2^{-2/3} (1.75 \mu K) \]

\[ = 1.10 \mu K \]

\( C_V \) is an increasing function of \( T \) since \( 1.3 \mu K \) is less than \( T_c \).