Last time: Renormalized $\phi^4$ theory:

\[ L = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 - \frac{\lambda}{4!} \phi^4 + \frac{1}{2} \delta_2 \phi \phi^2 \phi^2 - \frac{1}{2} \delta_m \phi^2 - \frac{\delta_3}{4!} \phi_1 \phi_1 \phi_1 \phi_1 \]

To find $\delta_2$ & $\delta_m$ we calculated $0$ using dim. reg.

Got: $\delta_2 = 0$ (on-shell, $\overline{\text{MS}}, \ms$)

\[ \delta_m^{\text{on-shell}} = \frac{2m^2}{32\pi^2} \left[ \frac{2}{\varepsilon} + \ln \left( \frac{\mu^2}{m^2} \right) - \zeta + \ln 4\pi + 1 \right] \]

\[ \delta_m^{\ms} = \frac{2m^2}{32\pi^2} \frac{2}{\varepsilon} \]

\[ \delta_m^{\overline{\text{MS}}} = \frac{\lambda m^2}{32\pi^2} \left[ \frac{2}{\varepsilon} - \zeta + \ln 4\pi \right] \]

To find $\delta_3$ we calculated $\chi$ using dim. reg.

$\Rightarrow$ got $\delta_3 = \frac{3\lambda^2}{16\pi^2} \frac{\mu^2}{\varepsilon} \frac{1}{3} + \text{finite}$

$\lambda$ depends on scheme

$\Rightarrow$ $\delta_3 = \lambda_0 \mu^2 \Rightarrow a \delta_3 = 0 \Rightarrow \delta_3 = 0 \Rightarrow \delta = 1 \Rightarrow$

$\lambda_0 = \delta_3 + \lambda \mu^2 \Rightarrow 0 = \mu^2 \frac{d}{d\mu^2} \lambda_0 = \mu^2 \frac{d}{d\mu^2} [\delta_3 + \lambda \mu^2]$

$\Rightarrow$ we found $\beta(\lambda) = \mu^2 \frac{d}{d\mu^2} \frac{\lambda}{\mu^2} = \frac{3\lambda^2}{32\pi^2}$

beta-function of $\phi^4$ theory!
The Callan–Symanzik Equation (cont'd)

\[ M = M(Q^2, \mu^2, \alpha_\lambda) = M \left( \frac{Q^2}{\mu^2}, \alpha_\lambda \right) \]

A dimensionless observable, depends only on one physical momentum scale \( Q^2 \).

\[ \mu^2 \frac{d}{d\mu^2} M \left( \frac{Q^2}{\mu^2}, \alpha_\lambda \right) = 0 \]

Independent of renormalization scale \( \mu \)

\[ \Rightarrow \left[ \mu^2 \frac{d}{d\mu^2} + \beta(\alpha_\lambda) \frac{d}{d\alpha_\lambda} \right] M \left( \frac{Q^2}{\mu^2}, \alpha_\lambda \right) = 0 \]

Callan–Symanzik equation 170

\[ \beta(\alpha_\lambda) = \mu^2 \frac{d\alpha_\lambda}{d\mu^2} \]

Beta-function
To solve the renormalization group (RG) equation define:

\[ \rho(\alpha_N) = \int_{\alpha_0}^{\alpha_N} \frac{d\alpha'}{\beta(\alpha')} \]

arbitrary cutoff

**Def.** Running Coupling by:

\[ d(Q^2) = S^{-1} \left( \ln \frac{Q^2}{\mu^2} + \rho(\alpha_N) \right) \]

\[ S^{-1} \sim \text{inverse function} \]

\[ \Rightarrow \text{note that} \]

(i) \[ d(\mu^2) = d_{\mu} \]

(ii) \[ \left[ \mu^2 \frac{2}{\partial \mu^2} + \beta(\alpha_N) \frac{2}{\partial \alpha_N} \right] d(\alpha^2) = 0 \]

Item (ii) is true because:

\[ \left[ \mu^2 \frac{2}{\partial \mu^2} + \beta(\alpha_N) \frac{2}{\partial \alpha_N} \right] \left( \ln \frac{Q^2}{\mu^2} + \rho(\alpha_N) \right) = -1 + \beta(\alpha_N) \frac{2\rho(\alpha_N)}{\partial \alpha_N} = 0 \]

\[ \frac{1}{\beta(\alpha_N)} \text{ by definition} \]

As \( M\left(\frac{Q^2}{\mu^2}, \alpha_N\right) \) does not depend on \( \mu \), we can put \( \mu = 0 \) and get:

\[ \mu^2 \rightarrow Q^2 \]

\[ M\left(\frac{Q^2}{\mu^2}, \alpha_N\right) = M\left(\frac{Q^2}{\mu^2}, \alpha(\mu^2)\right) = M\left(1, d(Q^2)\right) = M\left(d(Q^2)\right) \]

\[ \Rightarrow \text{any } M \text{ which is a function of } d(Q^2) \text{ only satisfies the RG equation} \]
Let's find $\alpha(\alpha^2)$ and prove that this is the same coupling that we had before: to find $\alpha(\alpha^2)$ all we need is the beta-function $\beta(\alpha)$. Usually in perturbation theory one gets:

$$\beta(\alpha) = \beta_2 \alpha^2 + \beta_3 \alpha^3 + \ldots$$

$\beta_2$ is scheme-independent.

$\beta_3$ depends on renormalization scheme (MS, \overline{MS}, "on-shell", etc.)

$\Rightarrow$ all higher $\beta_i$'s ($\beta_4, \beta_5, \ldots$) are also scheme-dependent.

In QED we showed that $\beta_2^{\text{QED}} = \frac{1}{3\pi}$.

For $\psi^4$ theory, if $\alpha \ll 1 \Rightarrow \beta_2^{\psi^4} = \frac{3}{32\pi^2}$.

Keep $\beta_2$ only ($\alpha \ll 1 \Rightarrow$ drop higher orders for now):

$$\beta(\alpha) = \beta_2 \alpha^2 \Rightarrow \beta(\alpha) = \frac{1}{\beta(\alpha) \beta_2} \int_{\alpha_0}^{\alpha} \frac{dx}{\alpha_x} = \frac{1}{\beta_2} \left( \frac{1}{\alpha_0} + \frac{1}{\alpha} \right) = \frac{1}{\beta_2} \left( \frac{1}{\alpha_0} + \frac{1}{\alpha} \right) \Rightarrow \beta(\alpha) = -\frac{1}{\beta_2} \left( \frac{1}{\alpha} - \frac{1}{\alpha_0} \right)$$

The inverse function: $\beta(\alpha) = \omega \Rightarrow \alpha = \beta^{-1}(\omega)$
\[ -\frac{1}{\beta_2} \left( \frac{1}{\alpha_r} - \frac{1}{\alpha_0} \right) = \delta \implies \frac{1}{\alpha} = \frac{1}{\alpha_0} - \beta_2 \delta \]

\[ \alpha = \beta^{-1}(\delta) = \frac{1}{\frac{1}{\alpha_0} - \beta_2 \delta} \]

\[ \alpha(Q^2) = \beta^{-1}\left(\frac{\alpha \alpha_0}{\alpha_0^2} + \beta(\alpha_0)\right) = \beta^{-1}\left(\frac{\alpha \alpha_0^2}{\mu^2} - \frac{1}{\beta_2} \left(\frac{1}{\alpha_r} - \frac{1}{\alpha_0}\right)\right) \]

\[ = \frac{\alpha_0}{1 - \beta_2 \alpha_0 \left(\frac{\alpha \alpha_0^2}{\mu^2} - \frac{1}{\beta_2} \left(\frac{1}{\alpha_r} - \frac{1}{\alpha_0}\right)\right)} \]

\[ = \frac{1}{\alpha_r - \beta_2 \frac{\alpha \alpha_0^2}{\mu^2}} \]

\[ \Rightarrow \alpha(Q^2) = \frac{d_{\mu}}{1 - d_{\mu} \beta_2 \frac{\alpha \alpha_0^2}{\mu^2}} \]

one-loop running coupling in a gauge theory

Cf. put \( \beta_{QED}^2 = \frac{1}{3 \pi} \) get QED running coupling

put \( \beta_{\psi^4}^2 = \frac{3}{32 \pi^2} \) and swap \( \alpha \leftrightarrow \lambda \) get \( \gamma \) running coupling.

QCD, \( \lambda \)

small \( Q^2 \) \( \Rightarrow \) large distances

when coupling is large, pert. theory is not reliable.

\( \alpha \) pert. theory works at small \( Q^2 \).
What does the full $\beta(x)$ may look like?

One possibility is this: $\beta(x)$

As $\mu^2 \frac{d \mu}{d \mu^2} = \beta(\mu) \Rightarrow$

$\Rightarrow \alpha_\mu^2 = \alpha_0 \Rightarrow \beta(\alpha_0) = 0$

$\Rightarrow \mu^2 \frac{d \mu}{d \mu^2} = 0 \Rightarrow \mu$ becomes a constant:

$\alpha(\alpha^2)$

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A "fixed point" coupling is "frozen" at $\alpha_0$.

$UV$ happens at large $\alpha^2$.

Renormalization group: general discussion.

Consider renormalized n-point Green function:

$$G^{(n)}(x_1, \ldots, x_n) = \langle \phi_0 | T \phi(x_1) \cdots \phi(x_n) | \phi_0 \rangle$$

\[ \text{physical fields.} \]

In momentum space get

$$G^{(n)}(p_1, \ldots, p_n) = \int d^4x_1 \cdots d^4x_n e^{i p_1 \cdot x_1 + \cdots + i p_n \cdot x_n} \cdot G^{(n)}(x_1, \ldots, x_n).$$
Renormalization "group" is defined by group transformations: \( \mu \to \lambda \mu \), \( \lambda = \text{constant} \). This is simply a rescaling of \( \mu \).

Physical observables \( \langle \cdots \rangle \) are \( \mu \)-independent \( \langle \cdots \rangle \to \langle \cdots \rangle \) independent of \( \mu \).

Now, in general \( \langle \cdots \rangle \) should not depend on \( \mu \):

\[
\mu^2 \frac{d}{d\mu^2} \left[ \frac{\hat{Z}^{(n)}}{\hat{Z}} \right](\mu, \cdots, \mu^n) = 0.
\]

However, \( G^{(n)} \) has in it functions of \( \mu \):
- \( \alpha_\mu \sim \text{coupling constant} \)
- \( m_\mu \sim \text{renormalized mass} \) (e.g. \( s_m = m^2 \hat{Z} \mu^{-2} \))

\[
(\phi(x) = \frac{1}{\sqrt{2}} \psi_0(x) \Rightarrow \hat{Z} \text{ depends on } \mu.)
\]

\[
\left. \begin{array}{l}
\hat{Z} \text{ physical field} \Rightarrow \langle \cdots \rangle = \langle \cdots \rangle \\
\text{bare field} \Rightarrow \langle \cdots \rangle \to \hat{Z}^{-1/2} G^{(n)}(\cdots)
\end{array} \right\} \text{ LO \& reduction, see 4.1}
\]

\[
\Rightarrow 0 = \mu^2 \frac{d}{d\mu^2} \left[ \frac{G^{(n)}}{\hat{Z}} \right] = \left[ \mu^2 \left( \frac{2}{dx^2} + \mu^2 \frac{d\alpha_\mu}{d\mu^2} \right) + \mu^2 \frac{d\alpha_\mu}{d\mu^2} + \mu^2 \frac{d}{d\mu^2} \frac{2}{\hat{Z} \mu^{-2}} \right] G^{(n)}.
\]

Define

\[
\beta(\alpha_\mu) = \mu^2 \frac{d\alpha_\mu}{d\mu^2} \sim \text{as before}
\]

\[
\gamma(\alpha_\mu) = \mu^2 \frac{d\ln \sqrt{\hat{Z}}}{d\mu^2} \sim \text{anomalous dimension}
\]

\[
\delta_m(\alpha_\mu) = \frac{1}{m^2} \mu^2 \frac{dm^2}{d\mu^2}.
\]
\[ \alpha_s \equiv \frac{n_c}{2} \mu^2 \frac{d \pi^{n-1}}{d \mu^2} = \frac{n_c}{2} \left( \frac{n_c}{2} \right)^2 \pi \\frac{d \alpha_s}{d \mu^2} = -\frac{n_c}{2} \frac{d \alpha_s}{d \mu^2} = \delta(\alpha_s) \]

\[ \Rightarrow \frac{n_c}{2} \mu^2 \frac{d \pi^{n-1}}{d \mu^2} = -\nu \delta(\alpha_s) \]

\[ \Rightarrow \left[ \left( \mu^2 \frac{d}{d \mu^2} + \beta(\alpha_s) \frac{d}{d \alpha_s} \right) - \nu \delta(\alpha_s) \right] \Gamma^{(\mu)} = 0 \]


\[ \Rightarrow \text{gives renormalization group (RG) flow of } \Gamma^{(\mu)} \]

other beta-functions: in QCD \[ \beta(\alpha_s) < 0 \] negative!
Example: Two-point function $G^{(2)}(p)$

$G^{(2)}(x_1, x_2) = \langle \Phi_0 | T \Phi(x_1) \Phi(x_2) | \Phi_0 \rangle = G^{(2)}(p) = \frac{i}{\mu^2} \cdot f(p^2, \mu^2)$

In general, the Callan-Symanzik eqn becomes:

$$\left[ \mu^2 \frac{2}{\partial \mu^2} + \beta(\alpha) \frac{2}{\partial \alpha} - 2 \delta(\alpha) \right] G^{(2)}(p) = 0$$

$\Rightarrow$ replace $\mu^2 \frac{2}{\partial \mu^2} \rightarrow -p^2 \frac{2}{\partial p^2} - 1 \Rightarrow$

$$\left[ p^2 \frac{2}{\partial p^2} - \beta(\alpha) \frac{2}{\partial \alpha} + 1 + 2 \delta(\alpha) \right] G^{(2)}(p) = 0$$

The solution is

$$G^{(2)}(p) = G_0^{(2)}(\alpha(p^2)) \cdot \frac{e^{- \int \frac{dp^2}{p^2} \left[ 1 + 2 \delta(\alpha(p^2)) \right]}}{\mu^2}$$

(checked!)

where $\alpha(p^2)$ is the physical coupling such that

$$\left[ p^2 \frac{2}{\partial p^2} - \beta(\alpha) \frac{2}{\partial \alpha} \right] \alpha(p^2) = 0 \quad \text{and} \quad \alpha(\mu^2) = \alpha_0.$$

If, for simplicity, we neglect running of the coupling

$\Rightarrow$ take $\delta(\alpha) \approx \frac{1}{2} C \alpha^2 \Rightarrow G^{(2)}(p) = G_0^{(2)} \cdot \left( \frac{p^2}{\mu^2} \right)^{-[1 + C \alpha^2]}$

$$G^{(2)}(p) \propto \left( \frac{p^2}{\mu^2} \right)^{-1 - C \alpha^2}$$
\[ 2 \int \frac{p^2}{p^1} \frac{dp^{12}}{p^{12}} \beta(\alpha_p) \frac{\partial}{\partial \alpha_p} \chi(\alpha(p^{12})) = 2 \int \frac{p^2}{p^{12}} \frac{\partial}{\partial \alpha_p} \delta(\alpha(p^{12})) \]

\[ = \frac{p^2}{p^{12}} \frac{\partial}{\partial \alpha_p} \]

\[ = 2 \delta(\alpha(p^1)) - 2 \delta(\alpha(p^{12})) \]

\[ \Rightarrow \text{makes it work out!} \]

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**Free Theory:** \( \beta = \chi = 0 \Rightarrow \frac{p^2}{p^{12}} \frac{\partial}{\partial \alpha_p} \chi(\alpha(p^{12})) = -\chi(\alpha(p^{12})) \)

\[ \Rightarrow \chi(\alpha(p^{12})) = \frac{i}{p^2} \quad \text{as expected.} \]