Regularization: a procedure to make divergent integrals finite in order to determine a part and a finite piece.

Two main types of regularization:

- **Pauli-Villars**: introduce "new" massive particles with masses $M_i^2 = m^2 + d_i \cdot M^2$. Each propagator becomes:

$$\frac{i}{k^2 - M_i^2 + i\epsilon} \rightarrow \sum_i c_i \frac{i}{k^2 - M_i^2 + i\epsilon}.$$  

We then take $M \rightarrow \infty$ limit keeping $\infty$ and finite terms only.

- **Dimensional regularization**: replace $4 \rightarrow d$ in the integrals, traces, ... Get some answer as a function of $d$, take $d \rightarrow 4$ limit ($\epsilon = 4 - d \rightarrow 0$) keeping divergent (e.g. $\frac{1}{\epsilon}, \frac{1}{\epsilon^2}, ...$) and finite (e.g. $\ln(4\pi)^{\frac{\epsilon}{2}}$, ...) terms.
We worked out the example of QED:

**Electron propagator corrections:**

\[ i \Sigma(p) = \text{sum of all 1PI graphs} \]

\[ -i \Sigma(p) = \frac{i}{p^2 - m_0^2 - \Sigma(p)} \]

\[ \text{got} \quad S(p) = \frac{i}{p^2 - m_0^2 - \Sigma(p)} \quad \text{dressed propagator}. \]

We want

\[ \frac{i}{p^2 - m_0^2 - \Sigma(p)} = -i \Sigma(p) + \left( \text{finite at } p^2 = m^2 \right) \]

\[ \text{bare mass} \uparrow \quad \text{physical mass} \downarrow \]

We calculated \( -i \Sigma_1(p) = \) (one-loop)

and found:

\[ S m = m - m_0 = \frac{3 \alpha_{em}}{4 \pi} m_0 \left\{ \ln\left( \frac{m^2}{m_0^2} \right) + \frac{1}{2} \right\} \]

\[ S \Sigma_2 = 2 \Sigma_2 - 1 = -\frac{\alpha_{em}}{4 \pi} \left\{ \ln\left( \frac{m^2}{m_0^2} \right) + \frac{9}{2} - 4 \int_0^1 \frac{dx}{1-x} \right\} \]

\[ S \Sigma_2 = -\frac{\alpha_{em}}{2 \pi} \frac{1}{\epsilon} + \text{const} \]

(we used Pauli-Villars regularization).

If we had used dim. reg.

**Photon propagator corrections:**

\[ i \Pi^{\mu \nu}(q) = \frac{i}{q^2} \Pi(q) \uparrow \quad \Rightarrow \quad \Pi^{\mu \nu}(q) = \left[ q^2 g^{\mu \nu} - g^{\mu \nu} q^2 \right] \Pi(q) \]

such that the dressed photon propagator is

\[ D^{\mu \nu}(q) = \frac{-i g^{\mu \nu}}{q^2 [1 - \Pi(q^2)]} + (g^{\mu \nu} \text{terms}) \]
Want \( D_{10}(q) = 2 \varepsilon \frac{-i g m}{q^2 + i \varepsilon} \Rightarrow 2 \varepsilon = \frac{1}{1 - \Pi(0)} \)

Used dim. reg. to find \( S_2^\varepsilon = \varepsilon - 1 = -\frac{\alpha}{3\pi} \left[ \frac{2}{3} \varepsilon - \delta(\varepsilon - \delta^2) \right] \)

\( \varepsilon = q - d. \)

**QED Vertex Correction:**

\[-i e \Gamma^\mu(p',p) = \begin{array}{c}
\text{Feynman sum} \\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\n
**Ward--Takahashi identity:**

\[-i q^\mu \Gamma^\mu(p',p) = S^{-1}(p) - S^{-1}(p') \]

Defining \( Z_1 \) by \( \lim_{q \to 0} \Gamma^\mu(p-q,p) \Rightarrow \frac{1}{Z_1} \)

Ward--Takahashi identity to prove that \( Z_1 = Z_2 \) in QED.

**Renormalization:** rearrangement of perturbation theory in such a way that at each order in the coupling all observables are finite.

For QED: start with bare Lagrangian:

\[ L_{QED} = \frac{e_0}{2} \left[ i \gamma \cdot A^0 + \frac{1}{4} F_{\mu \nu} F^{\mu \nu} - e_0 \bar{\psi} \gamma^\mu \psi A^\mu \right] \]

Define physical fields: \( \psi = \frac{1}{Z_2} \psi_0 \), \( A^\mu = \frac{1}{Z_3} A^\mu_0 \)

and the physical coupling: \( g = \frac{Z_2 Z_3^{1/2}}{Z_1} = e_0 \sqrt{Z_3} \)
One then has

\[ \mathcal{L} = - \frac{i}{4} F_{\mu \nu} F^{\mu \nu} + \overline{\psi} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma A_\mu \]

\[ - \frac{i}{4} S_3 F_{\mu \nu} F^{\mu \nu} + \overline{\psi} \gamma^\mu \gamma^{\nu} \gamma^\rho \gamma^\sigma A_\mu \]

Last line = *counterterms*.

\[ S_3 = Z_3 - 1, \quad S_2 = Z_2 - 1, \quad S_1 = Z_1 - 1, \quad S_m = Z_m - m. \]

New vertices in perturbation theory:

- \(-ie \gamma^\mu - \text{old vertex}\)

- \(-i \left[ q^2 g^{\mu \nu} - q^\mu q^\nu \right] S_3 \) \(\text{counterterms} \)

- \(i(\not{\! p} S_2 - S_m) \) \(\text{new vertices}\)

- \(-ie \gamma^\rho \)

How do we choose *counterterms*?

We want \( \overline{\psi} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma A_\mu \) to be finite.

However:

\[-i \Sigma(p) = -i \Sigma_2(p) + i(\not{\! p} S_2 - S_m) \]
In dim. reg. \[ \Sigma_2 (\rho) = \frac{-\alpha \varepsilon}{2\pi} \left\{ \frac{1}{\varepsilon} (\rho - y m) + \text{finite} \right\} \]

Check:

\[ S(\rho) = \frac{1}{\rho - y m - \Sigma(\rho)} = \frac{1}{\rho - y m + \frac{x}{2\pi} \left\{ \frac{1}{\varepsilon} (\rho - y m) \right\}} = \frac{1}{\rho \left(1 + \frac{x}{2\pi} \frac{1}{\varepsilon}\right) - y m \left(1 + \frac{\alpha}{\pi} \frac{1}{\varepsilon}\right)} \]

\[ = \frac{1}{1 + \frac{x}{2\pi} \frac{1}{\varepsilon}} \frac{1}{\rho - y m \left(1 + \frac{3x}{2\pi} \frac{1}{\varepsilon}\right)} \]

\[ \Sigma_2 = -\frac{x}{2\pi} \frac{1}{\varepsilon} + \text{finite} \]

\[ \Sigma_m = m \left(\frac{3x}{2\pi} \frac{1}{\varepsilon} + \text{finite}\right) \]

\[ \Sigma_2 = -\frac{x}{2\pi} \frac{1}{\varepsilon} + \text{finite} \]

\[ \Sigma_3 = -\frac{2x}{3\pi} \frac{1}{\varepsilon} + \text{finite} \]

\[ \Sigma_4 = \Sigma_2 \quad \text{Ward-Takahashi} \]

Problem: while requiring that the sum of diagrams is finite fixed the divergent parts of \( \Sigma_2, \Sigma_m \), it did not fix the constants (finite parts)! This is not a bug, but a feature: we are free to choose constants in any way we want! \( \Rightarrow \) different renormal schemes.
QED "on-shell" renormalization conditions:

\((i)\)  
$$\left. \Sigma(p) \right|_{p=m} = 0$$  
$$\left. \frac{\partial \Sigma(p)}{\partial p^2} \right|_{p=m} = 0$$

Want \( \Sigma(p) = \frac{i}{p-m} + \text{finite} \).

\((ii)\)  
$$\Pi(q^2=0) = 0$$  
want \( D_{\mu\nu}(q) = \frac{-i \gamma_\mu \gamma_\nu}{q^2 + i\epsilon} + (9g_{\mu\nu} - g_{\mu\nu}) \)

\((iii)\)  
$$\Gamma^\nu(q=0) = \gamma^\nu$$

\[\pi_m = -i e \gamma^5.\]

\(\Rightarrow\) 4 conditions fix \( S_1, S_2, S_3 \)  
& \(S_4\) uniquely!

We argued then that there is no other divergent 1-loop graphs in QED

\((e.g.\)  
\[\begin{array}{c}
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\end{array} \\
\end{array}\]
\(=0,\)

\(=0,\) ... Furry's theorem?

\(\mathcal{L}_{\text{QED}}\) is invariant under charge conjugation:

\[
\begin{align*}
\psi_x &\rightarrow C \bar{\psi}_\beta, \\
A_\mu &\rightarrow -A_\mu, \\
\psi &\rightarrow \gamma^\nu \psi
\end{align*}
\]

\(\bar{\psi} \gamma^\nu \psi \rightarrow -\bar{\psi} \gamma^\nu \psi \text{ (const)}\)

\(\Gamma_3 = \langle 0 | A_\mu(x) A_\nu(y) A_\rho(z) | 10 \rangle \rightarrow -\Gamma_3\), but as \(\mathcal{L}_{\text{QED}}\) is C-INV.

\(\Rightarrow -\Gamma_3 = (\Gamma_3)^C = \Gamma_3 \Rightarrow \Gamma_3 = 0.\)
In general can characterize the diagram by its superficial degree of divergence: \( D = 4N - P_e - 2P_\phi \)

\( L = \) \# loops \ (each loop gives \( d^4k \) )

\( P_e = \) \# of electron propagators \ (each fermion prop. gives \( 1/k \) )

\( P_\phi = \) \# \(-1\)-photon \(-1\)\ (each gives \( 1/4k^2 \) )

\( \Rightarrow \) the diagram should diverge at most as \( \Lambda^D \).

(If \( D < 0 \) \( \Rightarrow \) convergent diagram, Weinberg's thm)

\( L=1, P_e=6, P_\phi=0 \) \ (all other multi-leg 1-loops are finite too)

What about multi-loop graphs? One can show that UV divergences are removed by counterterms:

\[ \Rightarrow \] removed by \( \times \) counterterm at \( O(x^2) \)

\( \Rightarrow \) QED is renormalizable!
In general one can tell if the theory is renormalizable by dimension of the coupling constant: if \( \dim \lambda = \frac{1}{n} \)

\[ \Rightarrow \lambda \propto \frac{1}{\rho^n} \Rightarrow \text{each } \rho \text{ comes with } \frac{1}{\rho^n} \Rightarrow \text{get } \frac{(\frac{1}{\rho})^n}{n} \]

\[ \Rightarrow n > 0 \Rightarrow \text{higher orders have less UV divergences than lower orders} \]

\[ n = 0 \Rightarrow \text{higher order graphs are as divergent as lower order ones.} \]

\[ n < 0 \Rightarrow \text{higher order graphs are more divergent than lower ones.} \]

\[ \Rightarrow n > 0 : \underline{\text{super-renomalizable theory}} \]

\[ \quad (\text{e.g. } \phi^4 \text{ in } 3\text{-dim}) \]

\[ n = 0 : \underline{\text{renormalizable theory}} \]

\[ \quad (\phi^4 \text{ in } 4\text{-dim, QED}) \]

\[ n < 0 : \underline{\text{non-renormalizable theory}} \]

\[ \quad (\phi^6 \text{ in } 4\text{ dim.}) \]