Vertex Corrections and Ward-Takahashi identity

Def. \( \Gamma^\mu(p',p) = \tau_2 \rightarrow \Gamma^\mu(p',p) = \delta^\mu + \Lambda^\mu(p',p). \)

Showed that 
\[ -\frac{\partial \Sigma(p)}{\partial p^\mu} = \Lambda^\mu(p,p) \] 
Ward identity

or, equivalently, 
\[ -i \Gamma^\mu(p,p) = \frac{\partial}{\partial p^\mu} S^{-1}(p) \]

\[ S(p) = \frac{i}{p^2 - m - \Sigma(p)} \]

More generally 
\[ q^\mu \Lambda^\mu(p',p) = \Sigma(p') - \Sigma(p) \]

Ward-Takahashi identity

or 
\[ -i q^\mu \Gamma^\mu(p',p) = S^{-1}(p') - S^{-1}(p) \]

\[ q^\mu = p'^\mu - p^\mu \]

Def. 
\[ \lim_{q \to 0} \Gamma^\mu(\epsilon, p) = \frac{i}{\epsilon_1} \delta^\mu \rightarrow \text{using Ward-Takahashi identity we showed that} \quad \epsilon_1 = \epsilon_2. \]

Renormalization of QED (cont'd)

\[ L_{\text{QED}} = \bar{\psi}_0 [i\gamma^\mu - \gamma^5 m_0] \gamma_0 - i \gamma_5 F_{\mu
u} \psi_0 - e_0 \bar{\psi}_0 \gamma^\sigma A_\sigma \]

all fields and parameters are "bare".

Def. Dressed fields 
\[ \psi_0 = \sqrt{\frac{\epsilon_2}{2}} \gamma_0, \quad A_\sigma = \sqrt{\frac{\epsilon_2}{2}} A_\sigma, \]

physical coupling \( e \approx e_0 \epsilon_2 \epsilon_3^{1/3} \)

physical mass \( m \)
QED Lagrangian becomes:

\[
L_{\text{QED}} = -\frac{i}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^m \gamma^\lambda \epsilon_{\lambda \mu \nu \rho \sigma} \\
- \frac{1}{4} \delta_3 \epsilon_{\lambda \mu \nu \rho \sigma} \gamma^m \gamma^\lambda \epsilon_{\lambda \mu \nu \rho \sigma} \\
- \frac{1}{4} \delta_1 \epsilon_{\lambda \mu \nu \rho \sigma} \gamma^m \gamma^\lambda \epsilon_{\lambda \mu \nu \rho \sigma}
\]

where

\[
\delta_3 = \frac{2}{3} - 1, \quad \delta_2 = \frac{2}{2} - 1, \quad \delta_m = \frac{2}{2m_0 - m}, \\
\delta_1 = \frac{2}{1} - 1 = \frac{2}{e^2 \delta_3^{1/3} - 1} \quad \text{counterterms.}
\]

Feynman rules:

- For the "old" rules:
  \[
  \left( \frac{-i g_\rho}{q^2 + i\epsilon} \right) \frac{\gamma^\mu}{\rho} = \frac{i}{\rho - m}
  \]

- New vertices (counterterms):
  \[
  \begin{align*}
  &\bar{\psi} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^m \gamma^\lambda \epsilon_{\lambda \mu \nu \rho \sigma} \\
  &i (\bar{\psi} \gamma^\mu S_2 - \delta_m) \\
  &- i e S_1 \delta^\lambda
  \end{align*}
  \]
Field theory cannot predict particle masses or the coupling constant (it can predict momentum dependence of the coupling): these are external parameters. We can adjust bare parameters/counter-terms to make this work.

\textit{QED renormalization conditions: scheme}.

(i) We want that after renormalization the electron propagator is

\[ \frac{i}{p^2 - m^2} + \text{(terms regular)} \]

\[ \Rightarrow \frac{\Sigma(p)}{p^2 - m^2} = 0 \]

\[ \frac{\partial \Sigma(p)}{\partial p^2} \bigg|_{p=m} = 0 \]

\( n \text{ pole at } p^2 = m^2 \)

\( (\text{as } \Sigma(p) = \frac{i}{p^2 - m^2 - \Sigma(p)} ) \).

\( n \text{ residue } = i \text{ at } p = m \text{ pole.} \)

(\text{Remember, before renormalization we had})

\[ \frac{1}{\epsilon} = 1 - \frac{2\Sigma}{p^2} \bigg|_{p=m} \]

(ii) We want the photon propagator to be

\[ \frac{i}{q^2 + i\epsilon} = \frac{-i \eta_{\mu\nu}}{q^2 + i\epsilon} \]

\[ \Rightarrow \Pi(q^2 = 0) = 0 \]

\( n \text{ residue } = 1 \text{ at } q^2 = 0 \text{ pole.} \)

(iii) We want electron charge to be \( e \).

\[ -i e \delta^\mu \Rightarrow \Pi^\mu(q^2 = 0) = \delta^\mu \]
conditions (i) & (ii) fix $\delta_2, \delta_m, \delta_3$ with (iii) fixing $\delta_1$.

One-loop Structure & QED

Start with electron self-energy:

$$\Sigma_2(p) = -\text{i} e^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(p-k)^2 + i\varepsilon} \cdot \delta^p.$$ 

$$\frac{1}{\mu - m} \delta^p.$$ 

$$\Rightarrow -\text{i} \Sigma(p) = -\Sigma_2(p) + \text{i} (\not\partial \delta_2 - \delta_m)$$

$$\Rightarrow \Sigma(p) = \Sigma_2(p) - \not\partial \delta_2 + \delta_m$$

Calculate $\Sigma_2(p)$ using dim. reg.

$$\Sigma_2(p) = -\text{i} e^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(p-k)^2 + i\varepsilon} \frac{\delta^p (\not\partial + m) \delta^m}{k^2 - m^2 + i\varepsilon}.$$ 

In d-dimensions \{\delta^p, \delta^m\} = 2\delta^{m\nu}, \text{ stil' n} \Rightarrow \delta^p \delta_m = \delta^m \delta_p = d \delta^{p\nu} 

$$\delta^p \delta^m \delta^\nu \delta_m = \{\delta^p, \delta^m\} \delta^\nu \delta_m - \delta^p \delta^m \delta^\nu \delta_m = 2 \delta^\nu - d \delta^\nu =$$

$$= (2-d) \delta^\nu$$

$$\Rightarrow$$

$$\delta^\mu \delta^\nu = d$$

$$\delta^\nu \delta^\nu \delta^\mu = (2-d) \delta^\nu$$
\[ \Sigma_2(p) = -i e^2 \int \frac{d^4k}{(2\pi)^4} \frac{(2-d)k + dm}{[(p-k)^2 + i\epsilon][k^2 - m^2 + i\epsilon]} . \]

Introduce Feynman parameters & do with notation & integrate over momenta to get (cf. Peskin (10.41)):

\[ \Sigma_2(p) = \frac{e^2}{(4\pi)^{d/2}} \int_0^1 dx \frac{\Gamma \left( d - \frac{d}{2} \right) \left[ (2-d)x \phi + dm \right]}{\left[ (1-x)m^2 - x(1-x)p^2 \right]^{d/2}} \]

\[ \Rightarrow \left. \Sigma(p) \right|_{p^2=m^2, \ \not{p}=m} = 0 = \left. \Sigma_2(p) \right|_{p^2=m^2, \ \not{p}=m} - m \partial_2 + \partial_m \]

\[ \Rightarrow m \partial_2 - \partial_m = \frac{e^2}{(4\pi)^{d/2}} \int_0^1 dx \frac{\Gamma \left( d - \frac{d}{2} \right) \left[ 2x + d(1-x) \right] m}{\left[ (1-x)m^2 \right]^{d/2}} \]

\[ \frac{\partial \Sigma}{\partial \phi} \bigg|_{\phi=m} = \frac{\partial \Sigma_2}{\partial \phi} \bigg|_{\phi=m} - \partial_2 = 0 \]

\[ \Rightarrow \partial_2 = \frac{\partial \Sigma_2}{\partial \phi} \bigg|_{\phi=m} = \frac{e^2}{(4\pi)^{d/2}} \int_0^1 dx \frac{\Gamma \left( d - \frac{d}{2} \right)}{\left[ (1-x)m^2 \right]^{d/2}} \left\{ (2-d)x \right. \nonumber \]

\[ + \left( 2 - \frac{d}{2} \right) \left[ 2x m + (1-x) dm \right] \left[ (1-x)m^2 \right] \nonumber \]

\[ \times (1-x) 2m \} \]

\[ \Rightarrow \partial_2 \text{ and } \partial_m \text{ are fixed.} \]
Condition (ii) gives:

\[ \Delta^0 \Pi^\mu_2 (q) + \Pi^\mu_2 (q^2 - \delta_3) \]
\[ = \left[ \begin{array}{l}
\left( i \frac{\gamma^\mu}{q^2} \frac{\gamma^\nu}{q^2} \right) \Pi_2 (g^2) \\
 - \left[ \frac{q^2 \gamma^\mu - g^\mu g^\nu}{q^2} \right] \Pi_2 (g^2)
\end{array} \right] \]
\[ q^2 = 0 \]
\[ = \Pi (g^2) = \Pi_2 (g^2) - \delta_3 \Rightarrow \Pi (g^2 = 0) = 0 \] gives

\[ \delta_3 = \Pi_2 (g^2 = 0) = - \frac{d_{\text{ren}}}{3 \alpha} \left[ \frac{2}{3} - \delta + \lambda \varphi_3 - \lambda \varphi_3^2 \right] \]

Condition (iii) yields:

\[ \gamma^\mu + \gamma^\nu = \]
\[ -i e \Gamma^\mu (q) = -i e \left[ \Gamma_2^\mu (q) + \delta_1 \delta^\mu \right] \Rightarrow \text{want} \]
\[ \Gamma_2^\mu (q = 0) + \delta_1 \delta^\mu = \delta_1 \]
\[ \Rightarrow \frac{1}{2} \gamma^\mu = \frac{1}{2} \delta^\mu \Rightarrow \delta_1 = 1 - \frac{1}{2} \approx 2z - 1 = s \]
\[ \Rightarrow \delta_1 = \delta_2 \] as expected from Ward identity.

\[ \Rightarrow \text{fixed all counterterms: theory is renormalized at one loop.} \]

\[ \Rightarrow \text{one can show that there is no other one-loop divergences in QED:} \]
\[ \phi = 0, \quad \text{m} = \text{infrared}, \quad \gamma = 0 \] (Furry's)
In general can characterize the diagram by its superficial degree of divergence: \( D = 4L - P_e - 2P_\phi \)

\( L = \) # loops (each loop gives \( d^4 \chi \))

\( P_e = \) # of electron propagators (each fermion prop. gives \( 1/\chi \))

\( P_\phi = \) # -1- photon -1- (each gives \( 1/\chi^2 \)).

\( \Rightarrow \) The diagram should diverge at most as \( \Lambda^D \).

(If \( D < 0 \) \( \Rightarrow \) convergent diagram, Weinberg's theorem)

\( L = 1, \ P_e = 6, \ P_\phi = 0 \) (all other multi-leg 1-loops are finite too)

What about multi-loop graphs? One can show that UV divergences are removed by counterterms:

UV-div. only

\[ \frac{1}{\chi} \sim \frac{1}{\chi^2} \sim \frac{1}{\chi^3} \Rightarrow \text{finite} \]

\( L = 1, \ P_e = 6, \ P_\phi = 0 \) (all other multi-leg 1-loops are finite too)

\[ \Rightarrow \text{QED is renormalizable!} \]