\[ \text{Correction:} \quad \text{Cutoff rules as stated give } z i \text{Im} M. \]

Example: \( \Phi^3 \) Theory:

\[ \bar{c} M = (\pi \lambda)^2 \frac{\bar{c}}{\rho^2 - m^2 + i \varepsilon} = -i \frac{\lambda^2}{\rho^2 - m^2 + i \varepsilon} \]

\[ \Rightarrow M = -\frac{\lambda^2}{\rho^2 - m^2 + i \varepsilon} \Rightarrow 2i \text{Im} M = 2i \left( -\lambda^2 \right) \frac{1}{\rho^2 - m^2 + i \varepsilon} \]

\[ = i 2\pi S(\rho^2 - m^2), \lambda^2 \]

Using Cutoff rules: \( 2i \text{Im} M = (-i \lambda)^2 \frac{1}{\rho^2 - m^2 - i \varepsilon}. \)

\[ S(\rho^2 - m^2) \frac{1}{\rho^2 - m^2 - i \varepsilon} = i 2\pi S(\rho^2 - m^2), \lambda^2. \]

Inverting the "\( i \)" in \( i M. \)

Alternatively, one can modify Cutoff rules to give simply \( \text{Im} M \) by replacing rule (ii) with \( \frac{1}{\rho^2 - m^2 + i \varepsilon} \Rightarrow -\pi S(\rho^2 - m^2). \)
Field-Strength Renormalization: the Electron Self-Energy

Dressed electron propagator:

\[ S(p) = \frac{i}{\not{p} - m_0 - \Sigma(p)} \uparrow \Sigma_2 \frac{i}{\not{p} - m_{\text{phys}}} \]

\[-i \Sigma(p) \sim \text{sum of all 1PI diagrams.}\]

We calculated \[\begin{array}{c}
p \downarrow \\
\Sigma \\
\uparrow k \quad \text{using Pauli-Villars regularization to get} \end{array}\]

\[
\Sigma_2^{\text{reg}} (p) = \frac{\Delta_{\text{EM}}}{2\pi} \int_0^1 dx \left( 2m_0 - x \not{p} \right) \ln \left[ \frac{x M^2}{(1-x)m_0^2 - x(1-x)p^2} \right]
\]

Matching the poles required \( [\not{p} - m_0 - \Sigma_2(p)] \bigg|_{p = m_{\text{phys}}} = 0 \)

& get \( \Delta m = m_{\text{phys}} - m_0 = \Sigma_2(p) \bigg|_{p = m_0} + o(\Delta_{\text{EM}}^2) \).

This gave the mass shift:

\[
\Delta m = \frac{3\Delta_{\text{EM}}}{4\pi} m_0 \left\{ \ln \left( \frac{M^2}{m_0^2} \right) + \frac{1}{2} \right\}
\]

\( M \sim \text{UV regulator.} \)
To find the pole "residue" $\Sigma_2$, expand around it:

$$\phi - m_0 - \Sigma_2 (\phi) = \phi - m_0 - \Sigma_2 (\phi = m_{\text{phys}}) - \left. \frac{\partial \Sigma_2}{\partial \phi} \right|_{\phi = m_{\text{phys}}}.$$

Can think of as $\phi_{\text{fin}}$ of $\phi$.

$$(\phi - m_{\text{phys}}) + o((\phi - m_{\text{phys}})^2) =$$

$$= (\phi - m_{\text{phys}}) \left( 1 - \left. \frac{\partial \Sigma_2}{\partial \phi} \right|_{\phi = m_{\text{phys}}} \right) + \ldots$$

$$\Rightarrow \quad \frac{1}{\Sigma_2} = 1 - \left. \frac{\partial \Sigma_2}{\partial \phi} \right|_{\phi = m_{\text{phys}}}$$

$$\Rightarrow \quad \frac{1}{\Sigma_2} - 1 = - \left. \frac{\partial \Sigma_2}{\partial \phi} \right|_{\phi = m_{\text{phys}}} = - \frac{2}{\pi} \int_{0}^{x_{EM}} \frac{dx}{2x} \left( 2m_0 - x\phi \right).$$

$$\ln \left[ \frac{x M^2}{(1-x)m_0^2 - x(1-x)\phi^2} \right] \bigg|_{\phi = m_{\text{phys}}} = \frac{\alpha_{EM}}{2\pi} \int_{0}^{x_{EM}} dx.$$ 

Note:

$$\left\{ \ln \left( \frac{x M^2}{(1-x)^2 m_0^2} \right) + \frac{m_0 (2-x) \cdot \frac{-2m_0}{(1-x)^2 m_0^2}}{2m_0 + o(\alpha_{EM})} \right\} + o(\alpha_{EM}^2)$$

$$\Rightarrow \text{defining } S \Sigma_2 = \Sigma_2 - 1 \text{ we write}$$
can do most \( x \)-integrals:

\[
S Z_2 = -\frac{\alpha'\mu}{2\pi} \left\{ \frac{1}{2} \ln \left( \frac{M^2}{m_0^2} \right) + \frac{5}{4} - 1 - \int_0^1 dx \cdot x \cdot \frac{2}{1-x} \right\}
\]

\[
+ \frac{2}{1-x} + 2 \int_0^1 \frac{dx}{1-x}
\]

\[
S Z_2 = -\frac{\alpha'\mu}{2\pi} \left\{ \frac{1}{2} \ln \left( \frac{M^2}{m_0^2} \right) + \frac{5}{4} - 1 + 2 - 2 \int_0^1 \frac{dx}{1-x} \right\}
\]

\[
S Z_2 = -\frac{\alpha'\mu}{4\pi} \left\{ \ln \left( \frac{M^2}{m_0^2} \right) + \frac{1}{2} - 4 \int_0^1 \frac{dx}{1-x} \right\}
\]

The divergence in \( x \)-integral can be removed by introducing photon mass \( \mu \). The divergence comes from small momenta \( k \) infrared (IR) divergence (aka collinear divergence, comes from \( p^2 = k^2 \), \( p^2 = m_0^2 \), no quark recoil). Such divergences can be remedied by properly defining observables (e.g. no single-quark production \( x \)-section, need IR resolution scale/cutoff).
Both $S_{E1}$ and $S_m$ are UV-divergent! This is not surprising since we've calculated electron's self-energy (i.e., electron feeling its own Coulomb field) ⇒ UV-divergent.

⇒ OK: $S_m$ is infinite. Does it mean that $m_{phys}$ is infinite too? Only if $m_0$ is finite. But $m_{phys}$ is the only observable here: demand that $m_{phys}$ is finite (and equal to the measured electron's mass) ⇒ $m_0$ would depend on the cutoff $M$ to satisfy this requirement.

⇒ We'll rearrange perturbation theory to systematically replace $m_0$ with $m$. 
Vacuum Polarization

Let us now perform the resummation for the photon propagator. We need to sum graphs like

\[ \text{\includegraphics[width=0.2\textwidth]{graph}} \]

As usual start with one-particle irreducible diagrams:

\[ \frac{g}{\mu} \text{\includegraphics[width=0.2\textwidth]{graph}} = i \Pi^\mu_0(q) \]

(propagators not included).

\[ \Pi^\mu_0(q) = A(q^2) g^\mu_0 + B(q^2) q^\mu q^0 \]

on general grounds. Impose current conservation.

\[ g^\mu \Pi_\mu = 0 \implies A q^0 + B \cdot q^2 \cdot q^0 = 0 \implies B = -\frac{A}{q^2} \]

\[ \implies \Pi^\mu(q) = A(q^2) \left[ g^\mu_0 - \frac{g^\mu q^0}{q^2} \right] \]

assume no pde in \( \Pi(q^2) \) at \( q^0 = 0 \).

\[ \implies \text{write } \Pi^\mu(q) = \left[ \frac{q^2}{q^2} g^\mu_0 - \frac{g^\mu q^0}{q^2} \right] \Pi(q^2) \]

Summing all 1PI bubbles get:

\[ \frac{g}{\mu} \text{\includegraphics[width=0.2\textwidth]{graph}} = \text{\includegraphics[width=0.2\textwidth]{graph}} + \frac{g}{\mu} \text{\includegraphics[width=0.2\textwidth]{graph}} + \frac{g}{\mu} \text{\includegraphics[width=0.2\textwidth]{graph}} + \ldots \]

\[ -\frac{g q_0}{q^2} + \frac{g q_0}{q^2} \left[ g^2 g^0 - g^0 g^0 \right] \Pi(q^2) -\frac{g q_0}{q^2} + \ldots \]
\[
= -\frac{i \gamma^{\mu}}{q^2} + \frac{i \gamma^\nu}{q^2} \left[ S_{\nu} - \frac{q_{\nu} q_0}{q^2} \right] \Pi(q^2) + \ldots
\]

projector on direction \( \pm q^5 \)

\[
\Rightarrow \left[ S_{\nu} - \frac{q_{\nu} q_0}{q^2} \right] \left[ S_{\nu} - \frac{q_{\nu} q_0}{q^2} \right] = S_{\nu} - \frac{q_{\nu} q_0}{q^2}
\]

\( q_5 \times [...] = 0 \).

\[
\Rightarrow \text{get} \quad -\frac{i \gamma^{\mu}}{q^2} + \frac{i \gamma^\nu}{q^2} \left[ S_{\nu} - \frac{q_{\nu} q_0}{q^2} \right] \left( \Pi + \Pi^2 + \ldots \right)
\]

\[
= \frac{-i}{q^2 \left[ 1 - \Pi(q^2) \right]} \left[ \gamma^\mu - \frac{q_\mu q_0}{q^2} \right] + \frac{-i}{q^2} \left[ \gamma^\nu - \frac{q_\nu q_0}{q^2} \right] + \frac{-i}{q^2} \left[ \gamma^\nu - \frac{q_\nu q_0}{q^2} \right] \left[ \frac{1}{1-\Pi} - 1 \right]
\]

dressed photon prop.

\[
= \left( \frac{-i}{q^2 \left( 1 - \Pi(q^2) \right)} \left[ \gamma^\mu - \frac{q_\mu q_0}{q^2} \right] + \frac{-i}{q^2} \left[ \gamma^\nu - \frac{q_\nu q_0}{q^2} \right] + \frac{-i}{q^2} \left[ \gamma^\nu - \frac{q_\nu q_0}{q^2} \right] \right) = D^{\mu}(q)
\]

Usually combine photon propagator to some currents which are conserved \( \Rightarrow \gamma_5 \gamma^\nu \) terms are not important.

Write

\[
\frac{-i \gamma^{\mu}}{q^2 \left( 1 - \Pi(q^2) \right)} = Z_3 \frac{-i \gamma^\nu}{q^2} + \text{(multi-particle states)}
\]

\( \Rightarrow \quad Z_3 = \frac{1}{1 - \Pi(q^2 = 0)} \)

(One can prove that there is no mass shift.)
\[
\frac{e_0^2}{g^2} Z_3 = e_0^2 Z_3 \text{ can absorb photon field renormalization } Z_3 \text{ into the coupling constant } e_0
\]

**Def.** Physical charge \( e^2 = e_0^2 Z_3 \), \( e = e_0 \sqrt{Z_3} \).

One also has running coupling: \( \alpha_{EM}(g^2) = \frac{e^2(g^2)}{4\pi} \).

\[ \alpha_0 = \frac{e_0^2}{4\pi} \Rightarrow \text{ in general get} \quad \frac{e_0^2/4\pi}{g^2(1-\Pi(g^2))} = \frac{\alpha_0}{g^2(1-\Pi(g^2))} \]

\[ = \frac{\alpha}{g^2 \left[ 1-\Pi(g^2)+\Pi(0) \right]} = \frac{\alpha(g^2)}{g^2} \]

\[ \Rightarrow \quad \alpha(g^2) = \frac{\alpha}{1 - \left[ \Pi(g^2) - \Pi(0) \right]} \quad \text{running coupling constant} \quad (q^2 \text{- dependent}) \]

Let us calculate \( \Pi_{\mu\nu}(g) \) in perturbation theory:

- Fermion loop order \( e^2 \)

\[ \Pi_{\mu\nu}(g) = (-i e)^2 (-1) \int \frac{d^4k}{(2\pi)^4} \]

\[ = e^2 \sum_k \text{Tr} \left[ \gamma^\mu \frac{i}{k-m} \gamma^0 \frac{i}{k+g-m} \right] \]

\[ = e^2 \sum_k \frac{\text{Tr} \left[ \gamma^\mu (k+m) \gamma^0 (k+g+m) \right]}{(k^2-m^2+i\epsilon)(k^2+(k+g)^2-m^2+i\epsilon)} \]