Last time: Derived formulas for cross section:

\[
\frac{d\sigma}{d\Omega} = \frac{1}{2 \xi_1, 2 \xi_2, \xi_f} \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3 2 \xi_i} \left| M(k_1, k_2; \rho_1, ..., \rho_n) \right|^2 \delta^{(4)}(k_f + \sum_{i=1}^{n} p_i - \sum_{i=1}^{n} \rho_i)
\]

and for decay rate:

\[
\frac{d\Gamma}{d\Omega} = \frac{1}{2m} \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3 2 \xi_i} \left| M(k; \rho_1, ..., \rho_n) \right|^2 \delta^{(4)}(k - \sum_{i=1}^{n} \rho_i)
\]

in terms of the scattering amplitude \(M\).

Note: for particles with spin, etc. have to:

(i) average \(|M|^2\) over quantum state's of incoming particles

(ii) sum \(|M|^2\) over final state particles.

The LSZ Reduction Formula (cont'd)

How to find \(M\)? Consider 2\rightarrow2 process:

\[
\langle M_{2\rightarrow2} \rangle S^{(4)}(k_1, k_2; p_1, p_2) = \langle p_1, p_2 | \Pi(1, k_1, k_2) \rangle.
\]

\[
|k_1, k_2) = \frac{1}{Z} \hat{a}_{k_1}^+ \hat{a}_{k_2}^+ |0\rangle, \quad \langle p_1, p_2 | = \langle 0 | \hat{a}_{p_1} \hat{a}_{p_2}^+
\]

\(Z\) - a factor to remove self-dressing effects, not relevant for scattering.
Likewise \( |p_1, p_2\rangle = \frac{1}{2} \hat{a}_{p_1} \hat{a}_{p_2} |10\rangle \) since

\[\psi_n(x, t = \infty) = \psi_n(x, t = \infty) \quad \text{in the interaction picture with} \quad t_0 = \infty.\]

Consider the S-matrix:

\[\langle p_1, p_2 | S | k_1, k_2\rangle = \frac{1}{2^2} \langle 0 | \hat{a}_{p_1} \hat{a}_{p_2} U(+\infty, -\infty) \hat{a}^{\dagger}_{k_1} \hat{a}^{\dagger}_{k_2} | 10\rangle.\]

Write \( \hat{a}^{\dagger}_{k_1} = \int d^3x \psi_I(x) i \delta_0 e^{-ik_1 \cdot x} \) \( \psi_I(x) \) here is in the interaction picture with \( t_0 = -\infty.\)

\[\langle p_1, p_2 | S | k_1, k_2\rangle = \frac{1}{2^2} \int d^3x \langle 0 | \hat{a}^{\dagger}_{p_1} \hat{a}_{p_2} U(+\infty, -\infty) \hat{a}^{\dagger}_{k_1} \hat{a}^{\dagger}_{k_2} | 10\rangle\]

\[\psi_I(x, x^0) \hat{a}^{\dagger}_{k_2} | 10\rangle \leftarrow \delta_0 e^{-ik_2 \cdot x} = \begin{bmatrix} \text{as the expression is} \\ \text{\(x^0\)-independent \(\Rightarrow x^0 \rightarrow -\infty\)} \end{bmatrix}\]

\[\begin{align*}
\frac{1}{2^2} \lim_{x^0 \rightarrow -\infty} & \left[ \int d^3x \langle 0 | \hat{a}^{\dagger}_{p_1} \hat{a}_{p_2} U(+\infty, -\infty) \hat{a}^{\dagger}_{k_1} \hat{a}^{\dagger}_{k_2} | 10\rangle \\
& T\left\{ \psi_I(x) e^{-i\int dt H_2(t)} \right\} \right] \\
& \left( \text{true at \(x^0 \rightarrow -\infty\)} \right) \\
= & -\frac{1}{2^2} \int d^3x \left[ \langle 0 | \hat{a}^{\dagger}_{p_1} \hat{a}_{p_2} T\left\{ \psi_I(x) e^{-i\int dt H_2(t)} \right\} \hat{a}^{\dagger}_{k_2} | 10\rangle \right] \\
& \leftarrow \delta_0 e^{-ik_2 \cdot x} \right] + \frac{1}{2^2} \lim_{x^0 \rightarrow \infty} \left[ \int d^3x \langle 0 | \hat{a}^{\dagger}_{p_1} \hat{a}_{p_2} T\left\{ \psi_I(x) e^{-i\int dt H_2(t)} \right\} \hat{a}^{\dagger}_{k_2} | 10\rangle \right] \\
& i \delta_0 e^{-ik_2 \cdot x}
\end{align*}\]

\[\hat{a}^{\dagger}_{k_2} | 10\rangle \leftarrow \delta_0 e^{-ik_2 \cdot x} \psi_I(x) U(+\infty, -\infty)\]
The 2\textsuperscript{nd} term = \frac{1}{\varepsilon^2} \langle 0 | \hat{a}_{\rho_1} \hat{a}_{\rho_2} \hat{a}_{\mathbf{k}_1}^+ (\mathcal{H}(\infty, -\infty) \hat{a}_{\mathbf{k}_2}^+ | 0 \rangle \\
= \frac{1}{\varepsilon^2} (2\pi)^3 2 \mathcal{E}_{\mathbf{k}_1} \int \delta^3 \left( \mathbf{r}_1 - \mathbf{r}_2 \right) \langle p_2 | s\bar{\mathbf{r}} | h_2 \rangle + \delta^3 \left( \mathbf{r}_2 - \mathbf{r}_1 \right) \langle p_1 | s\bar{\mathbf{r}} | h_1 \rangle \right]

\Rightarrow \text{disconnected graphs, no scattering} \Rightarrow \text{drop from the cross section. We have}

\langle p_1, p_2 | S | h_1, h_2 \rangle = \text{disconnected terms} + \frac{-i}{\varepsilon^2} \int d^4x \bar{\psi}_0 \left[ \langle 0 | \hat{a}_{\rho_1} \hat{a}_{\rho_2} \right]

\cdot \left\{ \Psi_\Sigma (x) e^{-i \int_0^\infty dt \mathcal{H}_\Sigma (t)} \right\} \hat{a}_{\mathbf{k}_2}^+ | 0 \rangle \bar{\psi}_0 e^{-i \mathbf{k}_1 \cdot \mathbf{x}}

\left[ \mathcal{E}_0 \left[ f(x^0) \bar{\psi}_0 e^{-i \mathbf{k}_1 \cdot \mathbf{x}} \right] = -\left( \mathcal{E}_0^2 f(x^0) \right) e^{-i \mathbf{k}_1 \cdot \mathbf{x}} +

+ f(x^0) \bar{\psi}_0 e^{-i \mathbf{k}_1 \cdot \mathbf{x}} = -e^{-i \mathbf{k}_1 \cdot \mathbf{x}} \left[ \mathcal{E}_0^2 - \partial^2 + m^2 \right] f(x^0) =

- \mathcal{E}_0^2 = - (\mathbf{k}_1^2 + m^2) = - (\nabla^2 + m^2)

= \text{terms} = - (\nabla^2 + m^2)

= -e^{-i \mathbf{k}_1 \cdot \mathbf{x}} \left( \partial^2 + m^2 \right) f(x^0) \right).

\l\{ \bar{\psi}_0 \Psi_\Sigma (x) e^{-i \int_0^\infty dt \mathcal{H}_\Sigma (t)} \right\} \hat{a}_{\mathbf{k}_2}^+ | 0 \rangle \right]

\langle p_1, p_2 | S | h_1, h_2 \rangle = \text{disconnected terms} + \frac{-i}{\varepsilon^2} \int d^4x \bar{\psi}_0 e^{-i \mathbf{k}_1 \cdot \mathbf{x}} \left( \partial^2 + m^2 \right) f(x^0) \right).
Repeat the procedure for other $\hat{a}^+ \text{'s and } \hat{a} \text{'s: at the end have}

$$\langle p_1, p_2; \epsilon | R_1, k_2 \rangle = \text{disconnected terms} + \left(\frac{\hbar}{\sqrt{2}}\right)^4 \int d^3y_1 d^3y_2 d^3y_3 d^3y_4 \left\{ e^{-i \left( k_1 \cdot x_1 + k_2 \cdot x_2 + \epsilon p_1 \cdot y_1 + i p_2 \cdot y_2 \right)} \left( \Delta x_1 + m^2 \right) \left( \Delta x_2 + m^2 \right) \left( \Delta y_1 + m^2 \right) \left( \Delta y_2 + m^2 \right) \right\} \langle 0 | \prod \{ \psi^*_I(y_1) \psi^*_I(y_2) \psi^*_I(x_1) \psi^*_I(x_2) \} e^{-\frac{i}{\hbar} \int dt H_E(t)} \rangle \langle 0 |$$

Note that $\hat{a}_{2}^2 = \int d^3y \ e^{i k_2 \cdot y} \ e^{i \frac{\hbar}{2} \psi_I(y)} \ e^{i \frac{\hbar}{2} \psi_I(y)} \ e^{-i \frac{\hbar}{2} \psi_I(y)} \ e^{-i \frac{\hbar}{2} \psi_I(y)}$ are also independent.

$\Rightarrow$ for $\hat{a} \text{'s, take } y_0 \rightarrow +\infty \text{ to include } \psi^*_I(y) \text{ into time-ordering.}$

Finally, a small correction: our vacua are initially physical vacua, which we assume to be the same as perturbative vacuum at $t = -\infty$: $| \psi_0(-\infty) \rangle_I = | 0 \rangle = 1 | \psi_0 \rangle$, if $t_0 = -\infty$ in the interaction picture.

Similarly $\langle 0 |$ should be replaced by interaction picture vacuum at $t = +\infty$: $\langle 0 | \rightarrow \langle 0 | \psi_0(+\infty) \rangle_I = \langle 0 | \psi_0(+\infty) \rangle_I = \sum_n \langle 0 | \ U(-\infty,+\infty) \ | n \rangle \langle n | \ U(+\infty,-\infty) \ | 0 \rangle \langle 0 | = \frac{1}{\langle 0 | \ U(+\infty,-\infty) \ | 0 \rangle} \langle 0 | \ U(+\infty,-\infty) \ | 0 \rangle$
This would give \((\text{Gell-Mann-Low})\)

\[
\left< 0 \right| \mathcal{T} \left\{ \psi_1(y_1) \psi_2(y_2) \psi_1(x_1) \psi_2(x_2) e^{i \oint dt \mathbf{H}_c(t')} \right\} (0) \right> = \\
\left< 0 \right| \mathcal{T} e^{-i \oint dt \mathbf{H}_c(t')} (0) \right>
\]

\[
= \left< \psi_0 \right| \mathcal{T} \{ \psi_1(y_1) \psi_2(y_2) \psi_1(x_1) \psi_2(x_2) \} \left| \psi_0 \right> \text{.}
\]

In the end we get Lehmann, Symanzik & Zimmermann (LSZ) reduction formula (1955):

\[
\left< p_1, p_2 \right| S \left| h_1, h_2 \right> = \quad \text{disconnected terms} + \left( \frac{i}{\sqrt{2}} \right)^4 \int d^4 x_1, d^4 x_2, d^4 y_1, d^4 y_2 \quad \text{e}^{-i \mathbf{p}_1 \cdot \mathbf{x}_1 - i \mathbf{p}_2 \cdot \mathbf{x}_2 + i \mathbf{p}_1 \cdot \mathbf{y}_1 + i \mathbf{p}_2 \cdot \mathbf{y}_2} \quad \left( \mathcal{O}_{y_1} + m^2 \right) \left( \mathcal{O}_{y_2} + m^2 \right) \left( \mathcal{O}_{x_1} + m^2 \right) \left( \mathcal{O}_{x_2} + m^2 \right) \quad \left< \psi_0 \right| \mathcal{T} \{ \psi_1(y_1) \psi_2(y_2) \psi_1(x_1) \psi_2(x_2) \} \left| \psi_0 \right>
\]

1 true for any # of external legs.
2 with minor modification is also true for fields with spin.
2 only connected part contributes to \(M\):

\[
\mathcal{M}_{\mathbf{p_1} \rightarrow \mathbf{p_2}} (2\pi)^4 \delta^4 (\mathbf{k}+\mathbf{k}_2-\mathbf{p}_1-\mathbf{p}_2) = \left( \frac{i}{\sqrt{2}} \right)^4 \int d^4 x_1, d^4 x_2, d^4 y_1, d^4 y_2 \quad \text{e}^{-i \mathbf{p}_1 \cdot \mathbf{x}_1 - i \mathbf{p}_2 \cdot \mathbf{x}_2 + i \mathbf{p}_1 \cdot \mathbf{y}_1 + i \mathbf{p}_2 \cdot \mathbf{y}_2} \quad \left( \mathcal{O}_{y_1} + m^2 \right) \left( \mathcal{O}_{y_2} + m^2 \right) \left( \mathcal{O}_{x_1} + m^2 \right) \left( \mathcal{O}_{x_2} + m^2 \right) \quad \left< \psi_0 \right| \mathcal{T} \{ \psi_1(y_1) \psi_2(y_2) \psi_1(x_1) \psi_2(x_2) \} \left| \psi_0 \right>
\]
Now we see why we wanted to calculate a time-ordered product.

$\Rightarrow$ In momentum space each factor of $i(\mathbf{p} + m^2)$ becomes

$$i(-k^2 + m^2) = \frac{k^2 - m^2}{i} = \left[\frac{i}{k^2 - m^2}\right]^{-1}$$

the inverse propagator.

$\Rightarrow$ These factors remove propagators from external lines $\Rightarrow$ "truncates" the amplitude.

Also note that outgoing & incoming particles are on mass shell: $k_1^2 = k_2^2 = p_1^2 = p_2^2 = m^2$.

$\Rightarrow$ To calculate an amplitude, calculate Feynman diagram without propagators on external legs and putting all external lines on mass shell! (truncated diagram)

Feynman Rules for Scattering Amplitudes in $\phi^4$ theory.

1. Each internal line gives

$$\frac{k}{k^2 - m^2 + i\epsilon} = \frac{i}{k^2 - m^2 + i\epsilon}$$

2. Each vertex gives $\lambda = -i\lambda$

3. Each external line gives 1.
4. Impose 4-momentum conservation at each vertex. Integrate over each independent (internal) momentum \( \frac{d^4k}{(2\pi)^4} \).

5. Divide by symmetry factors.

6. Keep connected diagrams only.

Example: Consider 2 \( \rightarrow \) 2 scattering in \( \phi^4 \) theory at order 7. The only contributing diagram is

\[ \begin{array}{c}
\vec{k}_1 \quad \vec{p}_1 \\rightarrow \\
\vec{k}_2 \quad \vec{p}_2
\end{array} \]

\( \rightarrow \) by the rules \( i M_{2 \rightarrow 2} = -i \lambda \).

The cross section is

\[
d\sigma = \frac{1}{2 \epsilon_{k_1} 2 \epsilon_{k_2}} \frac{d^3p_1}{(2\pi)^3 2\epsilon_{p_1}} \frac{d^3p_2}{(2\pi)^3 2\epsilon_{p_2}} \frac{1}{2!} |M|^2 (2\pi)^4 \delta^{(4)}(p_1 + p_2 - k_1 - k_2) \]

Consider center-of-mass frame,

\( (\vec{k}_1 = -\vec{k}_2) : |\vec{k}_1 - \vec{k}_2| = \frac{\epsilon_{k_1} - \epsilon_{k_2}}{\epsilon_{k_1}} = 2 \frac{|\vec{k}_1|}{\epsilon_{k_1}} \) as \( \epsilon_{k_1} = \epsilon_{k_2} = \epsilon_k \).

\[
d\sigma = \frac{1}{8 \epsilon_k |\vec{k}|} \frac{d^3p_1}{(2\pi)^3 2\epsilon_{p_1}} \frac{d^3p_2}{(2\pi)^3 2\epsilon_{p_2}} \frac{1}{2!} |M|^2 (2\pi)^4 S(\epsilon_{p_1} + \epsilon_{p_2} - 2\epsilon_k) \]

\[
S^{(4)}(\vec{p}_1 + \vec{p}_2) = \frac{1}{8 \epsilon_k |\vec{k}|} \frac{(2\pi)^2}{2!} \frac{1}{(2\pi)^2 - 4\epsilon_k^2} \frac{d^3p_1}{2} \delta(\epsilon_{p_1} - \epsilon_k)
\]

where \( \epsilon_{p_1} = \epsilon_{p_2} = \epsilon_p \) as \( \vec{p}_1 = -\vec{p}_2 \) due to \( S \)-function.
\[ d^3 p \; \delta (\varepsilon p - E_e) = d\sigma \cdot dp \cdot p^2 \delta (\sqrt{p^2 + m^2} - E_e) = \]
\[ d\sigma = \frac{1}{2!} \frac{1}{8 \varepsilon_e^2 (2\pi)^2} \frac{1}{8 \varepsilon_e^2 (2\pi)^2} d\varepsilon_e \]
\[ \left( \frac{d\sigma}{d\varepsilon_e} \right)_{\text{CMS}} = \frac{1}{256 \pi^2 \varepsilon_e^2} \]

It is useful to define Mandelstam variable
\[ s = (k_1 + k_2)^2 \]

\[ \Rightarrow s = (k_1 + k_2)^2 = 4 \varepsilon_e^2 \Rightarrow \sqrt{s} = 2 \varepsilon_e \]
\[ \Rightarrow \left( \frac{d\sigma}{d\varepsilon_e} \right)_{\text{CMS}} = \frac{1}{2} \frac{1}{64 \pi^2 s} \]
\[ \Rightarrow \text{For } \gamma^4 \text{ theory we had} \]
\[ 1M_1^2 = \lambda^2 \Rightarrow \]
\[ \Rightarrow \left( \frac{d\sigma}{d\varepsilon_e} \right)_{\text{CMS}} = \frac{\lambda^2}{64 \pi^2 s^2} \]

Finally, a prediction which can be verified experimentally!

Quantum Electrodynamics (QED): Tree-Level processes

\[ L_{\text{QED}} = \bar{\psi} \left[ i \gamma^\mu D_\mu - m \right] \psi - \frac{i}{4} F^\mu_\nu F^{\nu\mu} \]
\[ D_\mu = \partial_\mu + ieA_\mu \]

\[ \Rightarrow \text{first need to find Feynman Rules for fermions & vector fields.} \]