Last time: Showed that vector fields transform as 4-vectors under Lorentz transformations.

\[ A_\mu \rightarrow A'_{\nu(x')} = \Lambda^\mu_\nu A^\nu(x) \]
\[ A^\mu \rightarrow A'^\mu(x') = \Lambda^\mu_\nu A^\nu(x) \]

We showed that \( \gamma_4 \) is a scalar.

\[ \gamma_5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \]

\( \gamma_5 \gamma_4 \) is pseudoscalar (changes sign under \( P \)).

We showed that \( \gamma_5 \gamma_4 \) is a 4-vector!

\( \gamma_5 \gamma_4 \gamma_5 \gamma_4 \) is pseudovector.

Dirac Lagrangian density

\[ L = \bar{\psi} \left( i \gamma^\mu \partial_\mu - m \right) \psi \]

EOM: \( \frac{\delta L}{\delta \bar{\psi}} = 0 \Rightarrow \left( i \gamma^\mu \partial_\mu - m \right) \psi = 0 \)  

(Def.) \( \sigma_{\mu \nu} = \frac{i}{2} \left[ \delta^\mu_\nu, \delta^{\lambda}_\lambda \right] \)

\( \Rightarrow \psi(x) \rightarrow \psi'(x') = e^{-i \frac{\gamma^\mu \sigma_{\mu \nu}}{2} \partial_\nu} \psi(x) \)

under Lorentz transformations.
Dirac Lagrangian is
\[ L = \bar{\psi} [i \gamma^\mu \partial_\mu - m] \psi \]

Any symmetries? Yes, we have \( L \to L \) under
\( \psi \to e^{i \alpha} \psi, \quad \bar{\psi} \to e^{-i \alpha} \bar{\psi} \), \( \alpha \) a real number
\( \Rightarrow \delta L = 0 \Rightarrow \) remember we had for scalar fields

\[ \delta L = \sum_a \partial_\mu \left[ \frac{\delta \bar{\psi}_a}{\delta (\partial_\mu \psi_a)} \right] \]

\( \Rightarrow \) similarly for spinors \( \psi \) & \( \bar{\psi} \) we have

\[ \delta L = \partial_\mu \left[ \frac{\delta \bar{\psi}_a}{\delta (\partial_\mu \psi_a)} \delta \psi_a + \frac{\delta \bar{\psi}_a}{\delta (\partial_\mu \bar{\psi}_a)} \delta \bar{\psi}_a \right] = 0 \]

\( \Rightarrow \) get

\[ \partial_\mu \left[ \bar{\psi} \gamma^\mu \psi \right] = 0 \]

\( \Rightarrow \) get

\[ \partial_\mu \left[ \bar{\psi} \gamma^\mu \psi \right] = 0 \Rightarrow \]

\[ j^\mu = \bar{\psi} \gamma^\mu \psi \]

is a conserved current: \( \partial_\mu j^\mu = 0 \) explicitly.
In general can construct any bi-linear object $\bar{\psi} \Gamma \psi$, with $\Gamma$ a $4 \times 4$ matrix. Full basis with definite Lorentz-transform properties of $4 \times 4$ matrices is 

$$
\Gamma = \{ 1, \gamma^0, \gamma^5, \gamma^0 \gamma^5, \sigma^{\mu \nu} \}
$$

where $\sigma^{\mu \nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]$. 16 matrices.

One has:

<table>
<thead>
<tr>
<th>Bilinear</th>
<th>Transformation law</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{\psi} \psi$</td>
<td>scalar</td>
</tr>
<tr>
<td>$\bar{\psi} \gamma^5 \psi$</td>
<td>pseudo scalar</td>
</tr>
<tr>
<td>$\bar{\psi} \gamma^\mu \psi$</td>
<td>vector</td>
</tr>
<tr>
<td>$\bar{\psi} \gamma^\mu \gamma^5 \psi$</td>
<td>axial vector</td>
</tr>
<tr>
<td>$\bar{\psi} \sigma^{\mu \nu} \psi$</td>
<td>antisymmetric tensor</td>
</tr>
</tbody>
</table>

$\gamma^5 = \bar{\psi} \gamma^5 \psi$ is also a 4-vector (axial current).

Is it conserved? In fact $\partial_\mu j^{\mu 5} = 2im \bar{\psi} \gamma^5 \psi$

$\Rightarrow$ it is conserved only if $m=0$.

Energy-momentum tensor: $T_{\mu \nu} = \frac{8 \pi}{\delta(0^4)} \frac{\delta^\mu}{\delta(\phi^4)} \frac{\delta^\nu}{\delta(\phi^4)} \bar{\psi} \gamma^\nu \psi$.

- Good by analogy with scalar field.
We get
\[ T_{\mu \nu} = i \overline{\psi} \delta_{\mu} \partial_{\nu} \psi - g_{\mu \nu} \left[ \overline{\psi} (i \gamma^a \partial_a - m) \psi \right] \]

\[ \Rightarrow T_{\mu \nu} = \overline{\psi} \left[ i \delta_{\mu} \partial_{\nu} - g_{\mu \nu} \left( i \gamma^a \partial_a + g_{\mu \nu} \right) \right] \psi \]

However, we can simplify this by using Dirac equation \((i \gamma^a \partial_a - m) \psi = 0 \Rightarrow \) get

\[ T_{\mu \nu} = i \overline{\psi} \delta_{\mu} \partial_{\nu} \psi \] (not symmetric)

Remember that the Hamiltonian \( H = \int d^3x \ T_{00} \).

We get
\[ H = \int d^3x \ i \overline{\psi} \gamma^0 \partial_0 \psi = \int d^3x \ i \gamma^+ \partial_+ \psi \]

\[ \psi^+ \gamma^0 \gamma^+ \psi \]

\[ \frac{1}{1} \]

\[ \Rightarrow H = \int d^3x \ i \gamma^+ \partial_+ \psi \] problem: \( H \) is not \( \geq 0 \! \)

(This is different from scalar fields, for which \( H \) was \( \geq 0 \) for the field!)

\( T_{\mu \nu} \) can be symmetrized:
\[ T_{\mu \nu}^{\text{sym}} = i \overline{\psi} \left[ \frac{1}{2} \left( \delta_{\mu} \partial_{\nu} + \delta_{\nu} \partial_{\mu} \right) \right] \psi \]

we can show that \( \partial^\mu T_{\mu \nu}^{\text{sym}} = 0 \).

Here \( \frac{1}{2} \delta_{\mu} \psi = \frac{1}{2} \partial_{\mu} \psi = \partial_{\mu} \psi \).

Here \( \frac{1}{2} \delta_{\mu} \psi = \frac{1}{2} \partial_{\mu} \psi = \partial_{\mu} \psi \).
Useful 8-matrix formulas:

\[ \{ 8^i, 8^j \} = 2 \eta^{ij} \]
\[ (\bar{8}^i)^+ = \bar{8}^i, (\bar{8}^i)^+ = -\bar{8}^i \]
\[ \bar{8}^i \bar{8}^j + \bar{8}^j \bar{8}^i \]
\[ \bar{8}^5 = \bar{8}^0 \bar{8}^1 \bar{8}^2 \bar{8}^3, \quad (\bar{8}^5)^+ = \bar{8}^5 \]
\[ (\bar{8}^0)^2 = -(\bar{8}^i)^2 = 1, \quad (\bar{8}^5)^2 = 1. \]
\[ \{ \bar{8}^5, 8^R \} = 0 \quad \text{easy to see that } (\bar{8}^R)^2 = g^{R\mu} \]
\[ \text{no summation} \]

(Easy to check.)

Also \( 8^R 8^R = 4 \), \( 8^R 8^R 8^R = -2 \bar{8}^R \).

Finally, note that \( \bar{8}^5 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) in Weyl basis

\( \Rightarrow \) Def. \( P_L = \frac{1 - \bar{8}^5}{2}, \quad P_R = \frac{1 + \bar{8}^5}{2} \)

\( \Rightarrow \) \( P_L \psi = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = \begin{pmatrix} \chi_L \\ 0 \end{pmatrix} \equiv \psi_L \)

\( P_R \psi = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = \begin{pmatrix} 0 \\ \chi_R \end{pmatrix} \equiv \psi_R \)

Can check that \( P_L^2 = P_L, \quad P_R^2 = P_R, \quad P_L P_R = P_R P_L = 0 \).

(\( \psi_L \) = helicity \(-\frac{1}{2}\), \( \psi_R \) = helicity \( \frac{1}{2} \), more later.)
Take Dirac equation \[ i \gamma^\mu \partial_\mu - m \] \( \psi(x) = 0 \).

In momentum space \( \psi(x) = \int \frac{d^4p}{(2\pi)^4} e^{-i p \cdot x} \psi(p) \)

\[ (\gamma^\mu p_\mu - m) \psi(p) = 0 \]; If \( m = 0 \) \( \Rightarrow \gamma^\mu p_\mu \psi(p) = 0 \)

As \( \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) & \( \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \) \( \Rightarrow \)

\[ \Rightarrow \gamma^\mu p_\mu = \begin{pmatrix} 0 & p_0 - \vec{p} \cdot \vec{\sigma} \\ p_0 + \vec{p} \cdot \vec{\sigma} & 0 \end{pmatrix} \] \( \Rightarrow \) Dirac equation becomes:

\[ \begin{pmatrix} 0 & p_0 - \vec{p} \cdot \vec{\sigma} \\ p_0 + \vec{p} \cdot \vec{\sigma} & 0 \end{pmatrix} \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = 0 \]

(Def.) Helicity operator \( \h = \frac{\vec{p} \cdot \vec{\sigma}}{1^p} \) \( \Rightarrow \) for spin-\( \frac{1}{2} \) particles have \( \vec{s} = \frac{1}{2} \vec{\sigma} \) \( \Rightarrow \) \( \h = \frac{1}{21^p} \vec{p} \cdot \vec{\sigma} \).

Physical meaning: projection of spin on \( \vec{p} \) direction.

We get \( (p_0 - \vec{p} \cdot \vec{\sigma}) \chi_R = 0 \)

\[ (p_0 + \vec{p} \cdot \vec{\sigma}) \chi_L = 0 \]

Hence, as \( |1^p| = p_0 \) we get \( \h \chi_R = +\frac{1}{2} \)

\( \h \chi_L = -\frac{1}{2} \)
\[ \Rightarrow \chi_\text{R} \text{ has helicity } +1 \]

\[ \chi_\text{L} = 1 - 1 \]

\[ \Rightarrow \chi_\text{R} \text{ is called right-handed as } \]

\[ \begin{array}{c}
\vec{s} \\
\rightarrow
\end{array} \rightarrow \vec{p} \]

the spin is \( \perp \) to \( \vec{p} \). (hence \( h = +\frac{1}{2} \))

\[ \Rightarrow \chi_\text{L} \text{ is called left-handed as } \]

\[ \begin{array}{c}
\vec{s} \\
\leftarrow
\end{array} \rightarrow \vec{p} \]

the spin is \( \perp \) to \( \vec{p} \). (hence \( h = -\frac{1}{2} \)).
Poincare Group

Add space-time translations: \( x^\mu \to x'^\mu = x^\mu + a^\mu \)

**Def.** Poincare group is a group of Lorentz transformations \( \Lambda^\mu_\nu \) and translations \( a^\mu \):

\[
\Lambda^\mu_\nu x^\nu + a^\mu
\]

\( \Lambda^\mu_\nu \) = 6 parameters (boosts & rotations)

\( a^\mu \) = 4 parameters

Total = 10 parameters.

**Generators:**

\[
\begin{align*}
P_{\mu} &= i \partial_\mu \\
J_{\mu\nu} &= i \left[ x_\mu \partial_\nu - x_\nu \partial_\mu \right] + \delta_{\mu\nu}
\end{align*}
\]

\( P_\mu = i \partial_\mu \) \( P_\mu \) = generator of translations.

\( P_0 = i \partial_0 \) \( P_0 = i \partial_0 \) \( P^\mu = i \partial^\mu = -i \partial_\mu = -i \nabla_\mu \)

\( \rho = -i \nabla \) n 3-momentum operator

**Poincare algebra:**

\[
\begin{align*}
[ P_\mu, P_\nu ] &= 0 \\
[ P_\mu, J_{\rho\sigma} ] &= i \left( g_{\mu\rho} P_\sigma - g_{\mu\sigma} P_\rho \right) \\
[ J_{\mu\nu}, J_{\rho\sigma} ] &= \text{same as before}
\end{align*}
\]
Operator $P_{\mu} P^\mu$ commutes with everything

$\Rightarrow$ Casimir operator.

**Def.** Pauli-Lubanski vector $\omega^\mu = \frac{i}{2} \varepsilon^{\mu \nu \rho \sigma} P_{\nu} P_{\rho}$

$\Rightarrow$ can show that $\omega \cdot \omega = -\frac{1}{2} \varepsilon^{\mu \nu \rho \sigma} \varepsilon_{\nu \rho \sigma} = \Lambda (\text{tr} \, \omega \cdot \omega)$. (trace of particle $w_0 = -m \, \hat{s}$)

$\omega \cdot \omega$ is another Casimir operator.

Representations are:

1. $P_{\mu} P^\mu = m^2 > 0 \Rightarrow \omega_{\mu} \omega^\mu = -m^2 s (s+1)$
   
   $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

2. $P_{\mu} P^\mu = 0 \Rightarrow \omega_{\mu} \omega^\mu = 0$, $\omega_{\mu} P^\mu = 0$ (always)
   
   $\Rightarrow \omega \cdot \omega = h P^\mu \Rightarrow h = 0, \frac{1}{2}, 1, \ldots$ a helicity.

   In general helicity $h = \pm s$, $s$ is spin, discretized.

   (e.g. photon has 2 polarizations corresponding to $h = \pm 1$, neutrino has $h = \pm 1, \ldots$)

3. $P_{\mu} P^\mu = 0$ but $s$ in continuous. Possible, but not realized in nature.