1. Show explicitly that the time evolution kernel we derived in class for a free non-relativistic particle

\[ U_{\text{free}}(q; t; q', t') = \sqrt{\frac{m}{2 \pi i \hbar (t - t')}} e^{\frac{i m}{2 \hbar} \frac{(q - q')^2}{t - t'}} \]

is unitary.

**a** (5 pts) First show that

\[ \int_{-\infty}^{\infty} dq' \ U_{\text{free}}(q; t; q', t') U_{\text{free}}(q', t'; q'', t'') = U_{\text{free}}(q, t; q'', t''). \]

**b** (5 pts) Then demonstrate that

\[ \lim_{t \to t'} U_{\text{free}}(q; t; q', t') = \delta(q - q'). \]

Explain why the results of parts **a** and **b** prove unitarity of the time evolution kernel.

2. **a.** (5 pts) By explicitly expanding the exponentials on the left-hand-side and carrying out the Grassmann integrals show that the following relation holds

\[
\int d\bar{\chi}_1 d\chi_1 d\bar{\chi}_2 d\chi_2 \exp \left[ -\sum_{i,j} a_{ij} \bar{\chi}_i \chi_j \right] \exp \left[ \sum_k (\bar{\chi}_k \xi_k + \bar{\xi}_k \chi_k) \right] = (\det A) \exp \left[ \sum_{i,j} \xi_i A_{ij}^{-1} \xi_j \right]
\]

where \( \chi_i \) and \( \xi_j \) are Grassmann variables, and \( A \) is a 2 \times 2 matrix with elements \( a_{ij} \).

**b** (5 pts) Show that

\[ -i \frac{\partial}{\partial \xi} F = \chi F = F \chi \]

\[ i \frac{\partial}{\partial \xi} F = \bar{\chi} F = F \bar{\chi} \]

for the function

\[ F = \exp \left[ i (\bar{\xi} \chi + \bar{\chi} \xi) \right]. \]

Here \( \chi \) and \( \xi \) are Grassmann variables.
3. (10 pts) Consider a non-Abelian gauge theory with the gauge field $A^a_\mu$ and the Lagrangian

$$ \mathcal{L} = -\frac{1}{4} F^{a}_{\mu\nu} F^{a\mu\nu}. $$

Here

$$ F^{a}_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu $$

with $f^{abc}$ the structure constants of the gauge group $SU(N)$.

Write the equations of motion for this theory. If we define $J^a_\mu$ by

$$ \partial_\nu F^{a\nu\mu} = J^a_\mu $$

what is $J^a_\mu$ for the above Lagrangian?

4. (10 pts) When constructing gauge-invariant Lagrangian for the non-Abelian gauge field $A^a_\mu$ one may consider another Lorentz-invariant

$$ I = \epsilon^{\mu\rho\sigma\nu} F^{a}_{\mu\nu} F^{a}_{\rho\sigma}. $$

Show that this term can be written as a 4-divergence,

$$ I = \partial_\mu K^\mu $$

and find the 4-vector $K^\mu$ explicitly in terms of the field $A^a_\mu$. Why can not the invariant $I$ serve as the Lagrangian for the non-Abelian field?

(Hint: you may find the identity

$$ f^{abc} f^{cde} = \frac{2}{N} (\delta^{ac} \delta^{bd} - \delta^{ad} \delta^{bc}) + d^{ace} d^{bde} - d^{bce} d^{ade} $$

useful. Here the gauge group is $SU(N)$ and $d^{abc}$ is the absolutely symmetric object defined by

$$ d^{abc} = 2 \text{Tr} \left( T^a \{ T^b, T^c \} \right) $$

with $T^a$ the generators of $SU(N)$ in the fundamental representation.)