1. (15 pts) Imagine that the full non-perturbative beta-function of QED were

$$\beta(\alpha) = \frac{\alpha^2}{3 \pi} \left[ 1 - e^{1-\frac{1}{\pi}} \right].$$

Find the running QED coupling constant $\alpha(Q^2)$ for such beta-function. Sketch $\alpha(Q^2)$ as a function of $Q^2$. Find the UV fixed point and determine the large-$Q^2$ asymptotics of $\alpha(Q^2)$, i.e., find how it approaches the fixed point.

2. a. (15 pts) Consider a harmonic oscillator in a background of a time-dependent external force (source) $j(t)$. The Lagrangian is

$$L = \frac{1}{2} m q^2 - \frac{1}{2} m \omega^2 q^2 + q j(t).$$

Using quasi-classical method for evaluation of path integrals find the time-evolution (Feynman) kernel

$$U(q_f, t_f; q_i, t_i) = \langle q_f(t_f) | e^{-\frac{i}{\hbar} \hat{H}(t_f-t_i)} | q_i(t_i) \rangle_S = H(q_f, t_f | q_i, t_i)_H$$

$$= \int [\mathcal{D}q] \exp \left\{ \frac{i}{\hbar} \int_{t_i}^{t_f} dt \mathcal{L}(t) \right\}.$$

You may use the result derived in class for the harmonic oscillator without the external force, though this time you also need to find the classical action in terms of $j(t)$. In evaluating the classical action assume that $q_i = q_f = 0$, or, more specifically, require that $q_{cl}(t) = 0$ when $j(t) = 0$. When solving classical EOM you may find Fourier-integral decomposition

$$q(t) = \int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{-\frac{i}{\hbar} E t} q_E$$

useful.

b. (10 pts) Use the result of part a to show that the two-point function for the harmonic oscillator without the external force is given by

$$\langle 0 | \hat{q}(t_1) \hat{q}(t_2) | 0 \rangle = \frac{i \hbar^2}{m} \int_{-\infty}^{\infty} \frac{dE}{2\pi} \frac{e^{-\frac{i}{\hbar} E (t_1-t_2)}}{E^2 - \hbar^2 \omega^2 + i \epsilon}. \quad (1)$$
c. (10 pts) Re-derive the two-point function in Eq. (1) by using creation and annihilation operators. For the simple harmonic oscillator (without the external force) write

$$\hat{q}(t) = \sqrt{\frac{\hbar}{2m\omega}} \left[ \hat{a} e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t} \right]$$

and use commutation relations for $\hat{a}$ and $\hat{a}^\dagger$ ($[\hat{a}, \hat{a}^\dagger] = 1$, all other commutators are zero) to obtain Eq. (1).