Last time: defined color of quarks.

Write out the QCD (Quantum Chromodynamics) Lagrangian:

$$L_{QCD} = g^a f (i \gamma^\mu (\partial - \mathbf{m}_f) q^a_f - \frac{1}{4} F_{\mu
u} F^{\mu
u}$$

$$+ g \frac{g}{f} \mathbf{F}^a \mathbf{T}_i \mathbf{A}_\mu^i (T^c_i)_{ba} q^a_f$$

with $T^c_i$ the generators of $SU(3)$, $A^i_\mu$ gluon fields $\mathbf{i} = 1, \ldots, 8$.

Elements of Group Theory (cont'd)

We defined a group $G$: (i) $f \cdot g = h \in G$, (ii) $f(g h) = (f g) h$

(iii) $f e = e f = f$. (iv) $f^{-1}$.

- Abelian: $f g = g f$, non-Abelian $[f g] \neq 0$.

$U(N)$: $N \times N$ unitary matrices $\rightarrow$ unitary group

$SU(N)$: $-1 \rightarrow \Phi$ det $U = 1$. $\rightarrow$ special $-1$-

- Representation: $f \rightarrow D(f)$:

  (i) $D(e) = 1$

  (ii) $D(g_1, g_2) = D(g_1) D(g_2)$

  - $D$ matrices $\rightarrow$ dim. of representation is their size.
Take a group \( \mathbb{Z}_4 \): it has \( \{ e, g_1, g_2, g_3 \} \).

Our example \( \{ 1, e^{i \pi/2}, e^{i \pi}, e^{i 3\pi/2} \} = \{ D(e), D(g_1), D(g_2), D(g_3) \} \) is one of the many possible representations of \( \mathbb{Z}_4 \).

**Def.** Dimension of representation is the dimension of the space of \( D \)-matrices.

**Def.** Representation is called reducible if there is a matrix \( M \) such that

\[
M D(g) M^{-1} = \begin{bmatrix}
D_1(g) & 0 \\
0 & D_2(g)
\end{bmatrix} \quad \text{for all } g \in G.
\]

\( \Rightarrow D = D_1 \oplus D_2 \oplus \ldots \)

A representation is called irreducible if no such matrix \( M \) exists.

**Def.** For two groups \( G = \{ g_1, g_2, \ldots \} \), \( H = \{ h_1, h_2, \ldots \} \) define direct-product group \( G \times H = \{ (g, h) \} \) such that \( g_1 h_1 \cdot g_2 h_2 = g_1 g_2 \cdot h_1 h_2 \).

**Lie Groups**

Imagine a group \( G \) with elements smoothly dependent on a continuous set of parameters \( \xi_i \), \( i = 1, \ldots, N \) : \( g(\xi_i) \in G \).
Assume that \( g(x = 0) = e \) (the identity element).

For a representation of the group:

\[ D(x = 0) = 1. \]

Taylor expand \( D(x) \) near 0:

\[ D(s x) = 1 + i s x \cdot \vec{X} + \ldots = 1 + i \sum \vec{s x} \cdot \vec{X} \]

(summation over repeating indices assumed)

\( \vec{x} \) are called **generators** of the group.

\[ D(x) = D(s x) D(s x) \ldots D(s x) = \lim_{k \to \infty} \left( 1 + i \sum \vec{s x} \cdot \vec{X} \right)^k = \lim_{k \to \infty} \left( 1 + i \frac{\vec{s x} \cdot \vec{X}}{k} \right)^k = e^{i \vec{s x} \cdot \vec{X}}. \]

A group with elements depending smoothly on continuous set of parameters \( \vec{x}, i = 1, \ldots, n \), with generators \( \vec{X} \) is called a **Lie group**.

\[ D(\vec{x}) = e^{i \vec{x} \cdot \vec{X}} \] as \( D \) can be a matrix.

\( \vec{X} \) can be a matrix; therefore in general \( [\vec{X}_i, \vec{X}_j] \) does not have to be 0.
\[ D(\hat{x}) D(\hat{\beta}) = e^{i \hat{x} \cdot \hat{x}} e^{i \hat{\beta} \cdot \hat{x}} = e \]

is also a group element \( \Rightarrow e^{i \hat{x} \cdot \hat{x}} e^{i \hat{\beta} \cdot \hat{x}} = e \)

\( \Rightarrow \) can show that for this to work we need

\[ [X_a, X_b] = i f_{abc} X_c \]

Lie algebra

\( f_{abc} \) structure constants of the group

\( f_{abc} = - f_{bac} \).

\( f_{abc} \) are real for unitary representations (for hermitean \( X_a \)).

**Example**

Take the group \( SU(2) \): unitary \( 2 \times 2 \) matrices with \( \det = +1 \) \((UU^* = U^*U = 1, \det U = 1)\).

(Defining representation)

Using Pauli matrices we can define a representation of \( SU(2) \):

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

\( \Rightarrow D(\hat{\sigma}) = e^{i \frac{\hat{\sigma} \cdot \hat{\alpha}}{2}} \)

\( \hat{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \) \( \alpha \)-vector.

Rotations around \( \frac{\hat{\alpha}}{|\alpha|} \) axis by angle \( |\alpha| \).
\[ \sigma_i^+ = \overline{\sigma_i} \quad \text{(hermitean)} \implies \text{any } 2 \times 2 \]

unitary matrix with \( \det = 1 \) can be represented as:

\[ e^{i \frac{\sigma_i \cdot \sigma_j}{2}} \]

Check:

\[ U U^+ = e^{i \frac{\sigma_i \cdot \sigma_j}{2}} e^{-i \frac{\sigma_i \cdot \sigma_j}{2}} = 1 \]

\[ \det U = \det e^{i \frac{\sigma_i \cdot \sigma_j}{2}} = \left| \det e^{A} \right| = e^{\text{tr} A} = 1 \]

as \( \text{tr} \, \sigma_i = 0 \).

Thus there are \((2^2 - 1) = 3\) different \( n \times n \) traceless hermitean matrices \( \Rightarrow \{ \sigma_i \} \) use up all possibilities.

Generators:

\[ J_i = \frac{\sigma_i}{2} \implies D(2) = e^{i \frac{\sigma_i \cdot \sigma_j}{2}} \]

\( \Rightarrow \text{SU}(2) \) is a Lie group.

We know that

\[ [\sigma_i, \sigma_j] = 2i \epsilon_{ijk} \sigma_k \]

\( \Rightarrow \) generators of \( \text{SU}(2) \) form a Lie algebra with structure constants \( \epsilon_{ijk} \)

\[ \epsilon_{ijk} : \text{totally anti-symmetric Levi-Civita symbol}, \quad \epsilon_{123} = 1, \quad \epsilon_{ij2} = -\epsilon_{j2i} = \epsilon_{j2i} \ldots \]

\[ \epsilon_{112} = 0 \ldots \]
Another example: SU(3): \(3 \times 3\) unitary matrices with \(\det = +1\)

Define Gell-Mann matrices:

\[
\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (\text{cf. Pauli matrices})
\]

\[
\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},
\]

\[
\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\]

Normalization convention: \(\text{tr} [\lambda_i \lambda_j] = 2 \delta_{ij}\)

There are \(3^2 - 1 = 8\) +traceless hermitian matrices

\(\Rightarrow\) these should work.

Generators of SU(3): \(T^a = \frac{\lambda^a}{2}\)

\[
\Rightarrow [T^a, T^b] = i f^{abc} T^c, \quad \text{with structure constants } f^{abc}, \text{ which are anti-symmetric under the interchange of any two indices.}
\]

\(\Rightarrow SU(3)\) is a Lie group with the generator algebra given above.
\[
\begin{array}{cccc}
| a | b | c | f_{abc} \\
|---|---|---|---|
| 1 | 2 | 3 | 1 \\
| 1 | 4 | 7 | \frac{1}{2} \\
| 1 | 5 | 6 | -\frac{1}{2} \\
| 2 | 4 | 6 | \frac{1}{2} \\
| 2 | 5 | 7 | \frac{1}{2} \\
| 3 | 4 | 5 | \frac{1}{2} \\
| 3 | 6 | 2 | -\frac{1}{2} \\
| 4 | 5 | 8 | \sqrt{3}/2 \\
| 6 | 7 | 8 | \sqrt{3}/2 \\
\end{array}
\]

\[
f_{112} = 0 \ldots
\]

...all other \( f_{abc} \)'s can be obtained from this table.

Casimir operator commutes with all generators:

\[
\frac{r^2}{T} = T_1^2 + T_2^2 + \ldots + T_n^2 = \frac{n^2 - 1}{2n}
\]

\( \Rightarrow \) for \( \text{su}(2) \) it is \( \frac{3}{4} \)

for \( \text{su}(3) \) it is \( \frac{4}{3} \).

\[
D(\mathbf{A}) = e^{i \mathbf{A} \cdot \mathbf{T}}, \text{ with } \mathbf{A} = (A_1, A_2, \ldots, A_8)
\]

an 8-component vector.

**Jacobi Identity and the Adjoint Representation**

\( \Rightarrow \) go back to some general Lie group with the generators \( X_a \) obeying some Lie algebra\n
\[
[ X_a, X_b ] = i f_{abc} X_c
\]

One can then easily prove **Jacobi identity**.
\[ [X_a, [X_b, X_c]] + [X_b, [X_c, X_a]] + [X_c, [X_a, X_b]] = 0. \]

(prove this by using definitions of commutators)

\( \Rightarrow \) plug in the commutator of Lie algebra to write

\[ f_{bcd} f_{ade} + f_{abd} f_{cde} + f_{cad} f_{bde} = 0 \]

these relations are obeyed by structure constants of any Lie group, e.g. \( SU(n) \).

Define\( \) The generators in the adjoint representation by \( (t^a)_{bc} = -i f_{abc} \) \( \Rightarrow \) the above relation gives

\[ [t^a, t^b] = i f_{abc} t^c. \]

\( \Rightarrow \) they obey the Lie algebra too!

Def. \( D(A) = e^{i A^a t^a} \) gives the adjoint representation of Lie group.
Consider representation of \( SU(n) \) in terms of \( n \times n \) unitary matrices \( U \) \( (UU^+=1) \) with \( \det U = \pm 1 \). Matrices \( U \) can be thought of as linear operators acting on the \( n \)-dim vectors \( \alpha \in C^n \):

\[
\alpha_i \mapsto \alpha'_i = U_{ij} \alpha_j.
\]

**Def.** A **scalar product** \( \alpha_i^* b_c = \alpha \cdot b \) is invariant under \( SU(n) \) transformations:

\[
a_i^* b_c \rightarrow a'_i^* b'_c = U_{ij}^* a_j^* U_{ic} b_c = a_j^* U_{ij} U_{ic} b_c = a_j^* \underbrace{U_{ij} U_{ic}}_{= \delta_{jc}} b_c = a_j^* b_j = a_i^* b_c.
\]

**Def.** Introduce **upper indices**: \( a_i^* = a_i \), \( U_{ij} = U_{ij} \), \( U_{ji} = (U^*)_{ji} \):

\[
\Rightarrow \quad a_i \rightarrow a_i^* = U_{ij} a_j.
\]

\[
a_i \rightarrow a_i^* = U_{ij} a_j
\]

\[
\Rightarrow \text{ scalar product is } a_i^* b_c = \alpha \cdot b
\]

**Unitarity** \( U_{ik} U_{kj}^* = U_{ki}^* U_{kj} = U_{ki} U_{ji}^* = \delta_{ij} \equiv \delta_{ij} \).
Def. $a_i$'s form a basis for fundamental (defining) representation of $SU(n)$, denoted $\mathfrak{h}$.

$a_i$'s form a basis for conjugate representation $\overline{\mathfrak{h}}$.

$e^{a_{i_1} \cdots a_{i_p}} = U_{i_1}^{j_1} \cdots U_{i_p}^{j_p} U_{j_1}^{k_1} \cdots U_{j_p}^{k_p} a_{k_1} \cdots a_{k_p}$

e.g. $S^{ij}$ is invariant, so is Levi-Civita symbol $\epsilon^{i_1 \cdots i_n}$.

$\Rightarrow$ in general tensors form reducible representations of $SU(n)$.

$\Rightarrow$ to reduce them to irreducible representations, note that permutation operator commutes with all $U$'s: $P_{12} a^{i_1 \cdots i_p} = a^{j_1 \cdots j_p}$.

$\Rightarrow P_{12} a^{i_1 \cdots i_p} = P_{12} U_i^k U_{j_1}^{i_1} U_{j_2}^{i_2} a^{k_1 \cdots k_p} = U_i^k U_{j_1}^{i_1} U_{j_2}^{i_2} P_{12} a^{k_1 \cdots k_p}$

$\Rightarrow$ organize all tensors by eigenstates of $P_{12}$: they could be symmetric or anti-symmetric.
\[ a^i_d : \delta^{ij}_{\hat{d}} = \frac{1}{2} (a^i_d + a^j_{\hat{d}}), \quad A^i_d = \frac{1}{2} (a^i_d - a^j_{\hat{d}}) \]

\[ \Rightarrow p_{12} \delta^{ij}_{\hat{d}} = \delta^{ij}_{\hat{d}}, \quad p_{12} A^i_d = -A^i_d \]

What is this good for?

Take a product of two representations:

\[ a^i_d b^j = \frac{1}{2} (a^i_d b^j + a^j_{\hat{d}} b^i) + \frac{1}{2} (a^i_d b^j - a^j_{\hat{d}} b^i) \]

Take SU(3) for example: \( a^i_d \) is 3, \( a^j_{\hat{d}} b^i \) is 3 \( \otimes \) 3.

\[ \frac{1}{2} (a^i_d b^j + a^j_{\hat{d}} b^i) \text{ has 6 indep. components} \Rightarrow \text{decide which a basis for representation 6.} \]

\[ \frac{1}{2} (a^i_d b^j - a^j_{\hat{d}} b^i) \text{ has 3 indep. components} \]

\[ \frac{1}{2} \delta^{ij}_{\hat{d}} = \sum_{k} a^k_d b^k \]

\[ \text{L} \Rightarrow \text{it is 3} \]

\[ \Rightarrow \text{we showed that 3} \otimes \text{3} = 6 \oplus 3 \]

\[ a^i_d b^j = \left(a^i_d b^j - \frac{1}{3} \delta^{ij}_{\hat{d}} a^k b_k\right) + \frac{i}{3} \delta^{ij}_{\hat{d}} a^k b_k \]

+ traceless 3 \times 3 matrix \quad 1 (\text{a singlet}) \]

\[ \Rightarrow 8 \text{ d.o.f. freedom} \Rightarrow a_{\text{adjoint representation}} \]

\[ \Rightarrow 3 \otimes 3 = 8 \oplus 1 \]