Introduction

This course will cover Standard Model of Particle Physics. The Standard Model comprises strong and electroweak interactions.

(i) Electroweak interactions:

- leptons (spin $\frac{1}{2}$ particles)
  - $e$, $\mu$, $\tau$ - electron, muon, tau
  - $\nu_e$, $\nu_\mu$, $\nu_\tau$ - neutrinos

Interactions between them is mediated by gauge bosons:

- $W^+$, $W^-$, $Z$ - massive spin-1 particles
- $\gamma$ - photon - massless

(ii) Strong interactions:

- quarks (spin $\frac{1}{2}$) have 6 flavors:
  - $u$, $c$, $t$ - up, charm, top
  - $d$, $s$, $b$ - down, strange, bottom
Quarks also have 3 colors such that each quark of a given flavor comes in in 3 different colors.

Quarks interact by exchanging gluons: $g$ a gluon, spin-1, massless particle.

There are 8 gluon colors

Quarks & gluons combine into bound states like mesons ($q\bar{q}$) & baryons ($qqq$)

$\pi^\pm, \pi^0, K, \rho, \omega, \ldots$

$\rho, \eta, \Sigma, \Sigma^0, \Lambda, \Xi, \ldots$

Higgs boson (spin-0) yet to be discovered.

The Standard Model does not include gravity: the "fundamental" interactions:

- strong
- electric
- weak

\underline{Standard Model}

Standard Model depends on 16 (!) external parameters (quark masses, lepton masses, couplings, CKM matrix).
however SM is surprisingly robust; it described everything we know about strong & electroweak interactions up until 2003, when neutrino masses were discovered, indicating that there is physics beyond SM.

Theories beyond SM have been proposed ever since the construction of SM, and include technicolor, supersymmetry, etc. (no exp. evidence yet)

A complete "theory of everything" should indeed incorporate (quantum) gravity & string theory is a possibility.

Nowadays, a lot of SM physics is considered "nuclear physics", while beyond SM physics is labelled "particle physics".
The theoretical language of SM is Quantum Field Theory (QM + special relativity).
Hence knowledge of QFT is needed for the course. We will start by reviewing some QFT material.
4-vectors, notations

Defining $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$, write Lorentz transformation as

$$
\begin{pmatrix}
  x'^0 \\
  x'^1 \\
  x'^2 \\
  x'^3
\end{pmatrix}
= \begin{pmatrix}
  \gamma & -\beta \gamma & 0 & 0 \\
  -\beta \gamma & \gamma & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
  x^0 \\
  x^1 \\
  x^2 \\
  x^3
\end{pmatrix}
$$

$\beta = \frac{v}{c}$

$\gamma = \frac{1}{\sqrt{1 - \beta^2}}$

**Definition**
A 4-vector $A^m = \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix}$ is an object which under Lorentz transformation transforms as

$$
\begin{pmatrix}
  A'^0 \\
  A'^1 \\
  A'^2 \\
  A'^3
\end{pmatrix}
= \begin{pmatrix}
  \gamma & -\beta \gamma & 0 & 0 \\
  -\beta \gamma & \gamma & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
  A^0 \\
  A^1 \\
  A^2 \\
  A^3
\end{pmatrix}
$$

(example: $x^m$

$A'^0$ is a contravariant vector: $A'^0 = \frac{\partial x'^0}{\partial x^0} A^0$

$B_m$ is a covariant vector: $B^m = \frac{\partial x^0}{\partial x'^m} B^0$

$= \frac{\partial y}{\partial x^n} \equiv \partial_n y$ with $y$, scalar field is a covariant vector
\[ \frac{\partial \Phi}{\partial x^\mu} = \frac{\partial x'^\nu}{\partial x^\mu} \frac{\partial \Phi}{\partial x'^\nu}. \]

**Tensors:** \( A^\mu B_\nu \) - contravariant, \( A_\mu B^\nu \) - covariant (rank 2), can have higher ranks.

**Def.** Scalar (inner) product of 2 vectors is \( A_\mu B^\mu \).

(assume summation)

It is Lorentz-invariant:

\[ A_\mu B^\mu = \frac{\partial x^\alpha}{\partial x'^\mu} A_\alpha \frac{\partial x'^\mu}{\partial x^\beta} B_\beta = \frac{\partial x^\alpha}{\partial x^\beta} A_\alpha B_\beta = A_\alpha B^\alpha. \]

**Def.**

The interval \[ ds^2 = dx^\mu dx^\nu. \]

It's a Lorentz-invariant too.

**Def.** The metric tensor \( g_{\mu\nu} \) is defined by

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \]

In our Minkowski space \( g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \)

(throughout the course we'll use this notation)

\( dx^\mu dx^\nu \) is also a Lorentz-scalar \( \Rightarrow dx^\mu = g_{\mu\nu} dx^\nu \)

\( \Rightarrow g_{\mu\nu} \) lowers 2 indices!
Example: \( x^\mu = (ct, \vec{x}) \Rightarrow x_\mu = g_{\mu\nu} x^\nu = (ct, -\vec{x}) \)

contravariant

In general \( A_\mu = g_{\mu\nu} A^\nu \), \( A^\mu = g^{\mu\nu} A_\nu \)

where \( g^{\mu\nu} \) is defined by requiring that

\( g^{\mu\nu} g_{\alpha\beta} = \delta^{\mu\alpha} \), i.e. that is trace = 0:

with \( A_\mu = g_{\mu\nu} A^\nu \Rightarrow g^{\alpha\mu} A_\mu = g^{\alpha\mu} g_{\mu\nu} A^\nu = \delta^{\alpha\nu} A^\nu = A^\alpha \Rightarrow A^\alpha = g^{\alpha\mu} A_\mu \).

\[
g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}
\]

Def: \( \partial_\mu \equiv \frac{2}{ct^2} \frac{\partial}{\partial \xi^\mu} \), \( \partial^\mu \equiv \frac{2}{ct^2} \frac{\partial}{\partial \xi^\mu} \Rightarrow \partial_\mu \xi \) is a covariant vector,

\( \partial^\mu \xi \) is a contravariant vector. (check !)

\( \partial_\mu A^\mu \) is a Lorentz - invariant.

\( \partial_\mu \partial^\mu = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \) is also Lorentz - invariant.

Examples: other important 4-vectors are

\[
p^\mu = \begin{pmatrix} \xi/c \\ \vec{p} \end{pmatrix}, \quad p^\mu = \begin{pmatrix} \xi/c \\ -\vec{p} \end{pmatrix} \Rightarrow p_\mu p^\mu = \left( \frac{\xi}{c} \right)^2 - \vec{p}^2 = m^2 c^2.
\]
\( A^\mu = (\Phi, \vec{A}) \) in \( E \times M \), \( \Phi \) an electric potential, 
\( \vec{A} \) - vector potential.

\[ J^\mu = (\rho, \vec{j}) \] with \( \rho \) the charge density, \( \vec{j} \) the current density.

**Notation:** from now on \[ c = 1 \] and \[ \hbar = 1 \]

"natural units":

\[ \Rightarrow \text{mass, momentum, energy are measured in the same units (eV, keV, MeV, GeV,...)} \]

\[ 1 \text{eV} = 1.6 \times 10^{-19} \text{J} \]

**Distances, time are measured in femto-meters also fermi (fm):**

\[ 1 \text{fm} = 5 \text{ GeV}^{-1} \]

\[ 1 \text{GeV} = 10^9 \text{ eV} \]

1 femto = \( 10^{-15} \) m. = 1 fm.

Proton's mass \( m_p = 0.938 \text{ GeV} \approx 1 \text{ GeV} \)

Electron's mass \( m_e = 0.511 \text{ MeV} = 0.5 \times 10^{-3} \text{ GeV} \)
Free Scalar Field (real)

\[ \phi(x^\mu) = \phi(x^0, \vec{x}) \] \text{ a function of space-time points } x^\mu.

**Classical Field Theory.**

In classical mechanics one has point particles \( i = 1, \ldots, N \) with the Lagrangian \( L(\dot{q}_i, \ddot{q}_i, t) \) and the action \( S = \int dt \ L(\dot{q}_i, \ddot{q}_i, t) \).

\( q_i \) = degrees of freedom (e.g. particle coordinates)

\( \dot{q}_i = \frac{dq_i}{dt} \) = generalized velocities

Now, instead of discrete point particles we have a field \( \phi(x, t) \)

<table>
<thead>
<tr>
<th>Classical Mechanics</th>
<th>Classical Field Theory</th>
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<tbody>
<tr>
<td>( q_i )</td>
<td>( \phi(x^0, \vec{x}) )</td>
</tr>
<tr>
<td>( i )</td>
<td>( \vec{x}, t )</td>
</tr>
<tr>
<td>( \dot{q}_i )</td>
<td>( \partial_{\mu} \phi ), ( \mu = 0, 1, 2, 3 )</td>
</tr>
<tr>
<td>( L(q_i, \dot{q}_i, t) )</td>
<td>( \int d^3x \ L(\phi, \partial_{\mu} \phi) )</td>
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</tbody>
</table>
$L$ is Lagrangian density. (usually called the lagrangian)

The action is $S = \int dt \, L = \int dt \, d^3x \, L (\varphi, \partial \varphi, \varphi) \overline{d^4x}$ (remember $c=1$)

$S$ is a Lorentz scalar (better be, physics is Lorentz invariant)

What about $d^4x' = dx^0 \, dx' \, dx^2 \, dx^3$? Remember that $x^' \, \nu = \Lambda^\mu_\nu \, x^\nu$ with $\Lambda^\mu_\nu = \frac{\partial x^' \, \nu}{\partial x^\mu}$ a matrix of $\mathbb{R}^{4 \times 4}$.

$\Rightarrow d^4x' = \det \Lambda \cdot d^4x$

Jacobian

Now, $\det \Lambda = \det \begin{pmatrix} 1 && -\beta \, x && 0 && 0 \\ -\beta \, x && 1 && 0 && 0 \\ 0 && 0 && 1 && 0 \\ 0 && 0 && 0 && 1 \end{pmatrix} = \delta^2(1-\beta^2) = 1$ (true in general)

$\Rightarrow d^4x' = d^4x \Rightarrow d^4x$ is a Lorentz scalar

$\Rightarrow L$ is a Lorentz scalar!

Just like in classical mechanics, in classical field theory dynamics is given by the least action principle; field $\varphi$ is determined by requiring that $S$ is stationary with respect to small perturbations around $\varphi$: $S[\varphi + \delta \varphi] = S'[\varphi] + o(\delta^2 \varphi^2)$.
\[ 0 = \frac{\partial}{\partial x^\mu} \int \frac{\partial L}{\partial (\partial_\mu \phi)} - \frac{L}{\partial \phi} + \frac{\partial L}{\partial (\partial_\mu \phi)} \delta \phi \] = (\text{as } \partial_\mu \phi = \partial_\mu \phi = \text{parxs}) = \int \frac{L}{\partial \phi} \delta \phi - \\
\partial_\mu \frac{\partial L}{\partial (\partial_\mu \phi)} \delta \phi \right] + \text{surface term} \]

\[ \Rightarrow 0 = \int \frac{L}{\partial \phi} \delta \phi \left[ \frac{L}{\partial \phi} \right] \text{ for any } \delta \phi \]

\[ \Rightarrow \left( \frac{\partial L}{\partial \phi} - \partial_\mu \frac{\partial L}{\partial (\partial_\mu \phi)} \right) = 0 \]

**Euler–Lagrange equations (aka equations of motion) for field \( \phi \).**

(EOM)

Now, \( \phi(x) \) is a scalar field \( \Rightarrow \) it is Lorentz-invar,

which means that: \( \phi(x) \rightarrow \phi'(x') = \phi(x) \)

\( \Rightarrow \) as \( x'^\mu = \Lambda^\mu_\nu x^\nu \) \( \Rightarrow \) \( x' = \Lambda \cdot x \) \( \Rightarrow \) \( \phi'(x) = \phi(\Lambda' x) \).

**Lagrangian density for massive scalar field:**

\[ L = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 \]

**EOM:**

\[ \frac{\partial L}{\partial \phi} = -m^2 \phi , \quad \frac{\partial L}{\partial (\partial_\mu \phi)} = \partial_\mu \phi \]

\[ \Rightarrow \partial_\mu \frac{\partial L}{\partial (\partial_\mu \phi)} = \partial_\mu \partial_\mu \phi \Rightarrow -m^2 \phi - \partial_\mu \partial_\mu \phi = 0 \]
\[ [\Box + m^2] \psi = 0 \quad \text{Klein-Gordon equation} \]

or \[ [\Box + m^2] \varphi = 0 \]

To solve K-G equation write \( \psi(x) = \int d^4k \ e^{-ik \cdot x} \tilde{\psi}(k) \)

with \( k \cdot x = k_\mu x^\mu = k^0 x^0 - \vec{k} \cdot \vec{x} \).

\[ [\Box + m^2] \psi = \int d^4k \ \tilde{\psi}(k) (\Box + m^2) e^{-ik \cdot x} = \int d^4k \ \tilde{\psi}(k) \]

\[ [-k^2 + m^2] \tilde{\psi} = 0 \quad \text{with} \quad k^2 = k_\mu k^\mu = (k^0)^2 - (\vec{k})^2. \]

\[ \Rightarrow [-k^2 + m^2] \tilde{\psi} = 0 \quad \Rightarrow \text{as} \quad \tilde{\psi} \neq 0 = \quad k^2 = m^2 \quad \text{or} \]

\[ E_k^2 - k^2 = m^2 \Rightarrow E_k = \pm \sqrt{k^2 + m^2} \quad \Rightarrow \text{define} \quad E_k = \sqrt{k^2 + m^2} \]

\[ \Rightarrow \psi(x) = \int \frac{d^3k}{(2\pi)^3 2E_k} \left[ a_k e^{-iE_k t + i\vec{k} \cdot \vec{x}} + a_k^* e^{iE_k t - i\vec{k} \cdot \vec{x}} \right] \]

most general solution.

Canonical Quantization

In your QM class you must have seen that if we treat K-G equation as the equation for a single-particle wave function \( \psi(x) \) (just like