For an EM wave: \( |\vec{k}| = k_0 = \frac{\omega}{c}, \ |\vec{k'}| = k'_0 = \frac{\omega'}{c} \)

\( \Rightarrow \) if the angle between \( \vec{k} \) and \( \vec{v} \) is \( \theta \) in \( k \)

and \( \theta' \) in \( k' \) \( \Rightarrow \) \( \omega' = \frac{\kappa}{\kappa'} \left( \omega - \beta \frac{\omega}{c} \cos \theta \right) \)

\( \Rightarrow \) \( \omega' = \kappa \omega \left( 1 - \beta \cos \theta \right) \) \( \text{Doppler shift} \)

\[
\begin{align*}
\frac{\omega'}{c} \cos \theta' &= \kappa \left( \frac{\omega}{c} \cos \theta - \beta \cdot \frac{\omega}{c} \right) \\
\frac{\omega'}{c} \sin \theta' &= \frac{\omega}{c} \sin \theta
\end{align*}
\]

\( \Rightarrow \) \( \cosh \theta' = \frac{\sin \theta}{\kappa (\cos \theta - \beta)} \) \( \text{(cf. with light alteration)} \)

Four-vectors.

We have seen one example: \( x^0 = ct, x^1 = x, x^2 = y, x^3 = z \)

\[
\begin{pmatrix}
    x_0' \\
    x_1' \\
    x_2' \\
    x_3'
\end{pmatrix}
= \begin{pmatrix}
    \kappa & -\beta \kappa & 0 & 0 \\
    -\beta \kappa & \kappa & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
    x_0 \\
    x_1 \\
    x_2 \\
    x_3
\end{pmatrix}
\]
Definition: A 4-vector $A^M$ is a set of 4 quantities $(A^0, A^1, A^2, A^3)$, which under Lorentz transformation transform as

$$
\begin{pmatrix}
A^0' \\
A^1' \\
A^2' \\
A^3'
\end{pmatrix} =
\begin{pmatrix}
1 & -\beta \delta & 0 & 0 \\
-\beta \delta & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
A^0 \\
A^1 \\
A^2 \\
A^3
\end{pmatrix}.
$$

$\Rightarrow$ $A^M$, $M = 0, 1, 2, 3$ is a contravariant vector if it transforms according to:

$$A^{'M} = \frac{\partial x^M}{\partial x'^N} A^N \quad \text{(equivalent to above)}$$

$\Rightarrow$ $B^M$, $M = 0, \ldots, 3$ is a covariant vector if

$$B^{'M} = \frac{\partial x^M}{\partial x'N} B^N.$$

Example: $\frac{\partial y}{\partial x^M}$ is a covariant vector as

$$\frac{\partial y}{\partial x^M} = \frac{\partial x^0}{\partial x'^M} \frac{\partial x^0}{\partial x^0} = \frac{\partial x^0}{\partial x'^M} \frac{\partial x^0}{\partial x^0}.$$

One can define tensors by

$$A^{'M} B^{'N} = \frac{\partial x^M}{\partial x'^a} \frac{\partial x^0}{\partial x'^0} A^a A^b = \Rightarrow \text{rank two contravariant tensor would be } C^{MN} = \frac{\partial x^M}{\partial x^a} \frac{\partial x^N}{\partial x^b} C^a_b, \text{ etc.}$$
Definition Scalar (inner) product of 2 vectors is defined by \( A_{\mu} \cdot B^\mu \) (summation assumed)

Let's prove that it's Lorentz invariant:

\[
A'_\mu \cdot B'^\mu = \frac{\partial x^\alpha}{\partial x'^\mu} A_{\alpha} \frac{\partial x'^\mu}{\partial x^\beta} B^\beta = \frac{\partial x^\alpha}{\partial x^\beta} A_{\alpha} B^\beta = \delta^{\alpha}_{\beta}.
\]

\( A_{\alpha} B^\alpha = A_{\alpha} B^\alpha \). Q.E.D.

The interval is a scalar: (it's Lorentz-invariant)

\[
ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2
\]

Define the metric tensor by

\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu
\]

\[
g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g_{\mu\nu} \quad \text{(Minkowski)}
\]

Note that \( dx_\mu dx^\mu \) is also a Lorentz-scalar.

Identifying \( dx_\mu = g_{\mu0} dx^0 \) we see that \( g_{\mu0} \) raises and lowers the indices.
\[ A^\mu = g^\mu_\nu A^\nu, \quad A^\nu = g^\mu_\nu A^\mu \]

\[ x^\mu = g^\mu_\nu x^\nu, \ldots \]

\[ g^\mu_\nu = g^{\mu_2 \cdot \nu_2} \cdot \delta_{v_1\nu_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = S^\mu_\nu \]

Indeed, as \[ A^\mu B^\nu = S^\mu_\nu A^\mu B^\nu = g^\mu_\nu A^\mu B^\nu \].

**Define** an abbreviated notation: \[ \partial_\mu = \frac{\partial}{\partial x^\mu} \]

\[ \partial^\mu = \frac{\partial}{\partial x^\mu} \]

\[ \Rightarrow \partial_\mu \phi \text{ is a covariant vector} \]

\[ \partial^\mu \phi \text{ is a contravariant vector (check!)} \]

\[ \partial_\mu A^\mu \text{ is Lorentz invariant} \]

Laplacian operator \[ \frac{\partial^2}{c^2 \partial t^2} - \nabla^2 = \partial_\mu \partial^\mu \]

is also Lorentz invariant.

\[ 4 \text{ - velocity} \]

Let's define a 4-vector for velocity:

\[ dx^\mu = (dx^0, dx^1, dx^2, dx^3) \Rightarrow v^\mu = \frac{dx^\mu}{dt} \]