\[ T^{\mu \nu} = \left( \begin{array}{cc} \text{energy density} & \text{momentum density} \\ \text{momentum density} & -\text{Maxwell's stress tensor} \end{array} \right) \]

\[ \partial_{\mu} T^{\mu \nu} = 0 \quad \text{(energy & momentum conservation)} \]

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**Radiation by Moving Charges**

Imagine a charge moving along some arbitrary trajectory. It gives rise to the current \( J^\mu \).

To find radiation by this charge, all we have to find is \( A^\mu \) from Maxwell equations

\[ \partial_\mu F^{\mu \nu} = \frac{4\pi}{c} J^\nu \]

Let's work in \( \partial_\mu A^\mu = 0 \) gauge \( \Rightarrow \Box A^\mu = \frac{4\pi}{c} J^\mu \).
First we need to find the Green function of the operator $\Box$:

$$\Box G(x, x') = \delta^4(x - x')$$

where $x, x'$ are 4-vectors $x_\mu, x'_\mu$.

The standard technique is a Fourier transform:

look for $G(x, x') = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-x')} \tilde{G}(k)$

(use translational invariance to argue that $G(x, x') = G(x-x')$).

Rewriting the eqn. for $G$ as $\Box G(x) = \delta^4(x)$

and recalling that $\delta^4(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot x}$

(here $k \cdot x = k_\mu x^\mu$) we get

$$-k^2 \tilde{G} = 1 \Rightarrow \tilde{G} = -\frac{i}{k^2}$$

$$\Rightarrow G(x) = -\int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot x} \frac{1}{k^2}$$

However, this is not the end of the story: write

$$G(x) = -\frac{1}{(2\pi)^4} \int d^3k \ e^{i k \cdot x_0} \int_{-\infty}^{\infty} dk_0 \ \frac{e^{-ik_0 x_0}}{k_0^2 - k^2}.$$
The $k_0$-integration has poles at $k_0 = \pm |k|$ just on the contour!

We have a freedom of moving the contour to go around the poles in any way we'd like.

We'll use two contour paths: 

and 

Other paths like 

are important in quantum field theory, but not in classical physics.

Here's our contours, which we label $r$ & $a$ for retarded & advanced.

First let's work out contour $r$:

$$\int_{-\infty}^{\infty} \, dk_0 \, \frac{e^{-ik_0 x_0}}{k_0^2 - k^2} = \begin{cases} 0, & \text{if } x_0 < 0 \text{ close in upper half-plane} \\ \pm i 2\pi, & \text{if } x_0 > 0 \text{ close in lower half-plane} \end{cases}$$

$$= \Theta(x_0) \cdot (-2\pi i) \left\{ \frac{1}{2|k|} e^{-i|k|x_0} - \frac{1}{2|k|} e^{i|k|x_0} \right\} = \Theta(x_0) \cdot (-2\pi i) \frac{1}{|k|^2} \left( -i \right) \sin(|k|x_0) = -2\pi \Theta(x_0) \frac{1}{|k|} \sin(|k|x_0).$$
The corresponding retarded Green function is

\[ G_{r}(x) = \frac{1}{(2\pi)^{4}} (-2\pi) \Theta(x_{0}) \int d^{3}k \ e^{i\frac{k^{2} - x^{2}}{2}} \frac{\sin(|k|x_{0})}{\frac{1}{2}k} = \]

\[ \frac{1}{(2\pi)^{3}} \Theta(x_{0}) \int_{0}^{\infty} dk \cdot k^{2} \cdot \int_{0}^{2\pi} d\phi \ e^{i\frac{k|x_{0}|\cos\theta}{k}} \frac{\sin(k|x_{0}|)}{k} = \]

\[ \frac{1}{2(2\pi)^{2}} \Theta(x_{0}) \int_{0}^{\infty} dk \cdot \sin(k|x_{0}|) \cdot 2 \cdot \sin(k|x|) = \]

\[ \frac{1}{2(2\pi)^{2}} \Theta(x_{0}) \int_{0}^{\infty} dk \ (e^{ikx_{0}} - e^{-ikx_{0}})(e^{ik|x|} - e^{-ik|x|}) = \]

\[ \frac{1}{2(2\pi)^{2}} \Theta(x_{0}) \int_{0}^{\infty} dk \ \delta(x_{0} + k|x|) - \delta(x_{0} - k|x|) \]

\[ \frac{1}{2(2\pi)^{2}} \Theta(x_{0}) \left\{ \int_{-\infty}^{\infty} dk \ e^{ik(x_{0} + 1|x|)} - \int_{-\infty}^{0} dk \ e^{-ik(x_{0} - 1|x|)} \right\} = \frac{1}{2(2\pi)^{2}} \Theta(x_{0}) \left\{ \delta(x_{0} + 1|x|) - \delta(x_{0} - 1|x|) \right\} \]

\[ \frac{1}{(2\pi)^{4}} \Theta(x_{0}) \delta(x_{0} - 1|x|) = \frac{1}{4\pi} \Theta(x_{0}) \delta(x_{0} - 1|x|) \]

\[ G_{r}(x) = \frac{1}{4\pi} \Theta(x_{0}) \delta(x_{0} - 1|x|) \]

Not obviously relativistically invariant. Noting that

\[ \delta(x^{2}) = \delta(x_{0}^{2} - 1|x|^{2}) = \frac{1}{2|x|} \left[ \delta(x_{0} - 1|x|) + \delta(x_{0} + 1|x|) \right] \]

write

\[ G_{r}(x) = \frac{1}{2\pi} \Theta(x_{0}) \delta(x^{2}) \]
\[ G_r(x-x') = \frac{1}{2\pi} \Theta(x_0 - x'_0) \delta((x-x')^2) \]

\[ (x-x')^2 \text{ is manifestly invariant} \]

\[ x_0 - x'_0 = c(t - t') \text{ & under boost} \]

\[ \Delta t = \delta \Delta \xi \text{ with } \delta > 0 \Rightarrow \Theta(\Delta t) = \Theta(\delta \xi) \]

\[ \Rightarrow \text{also invariant!} \]

The advanced Green function can be obtained using contour a, which yields:

\[ G_a(x-x') = \frac{1}{2\pi} \Theta(x'_0 - x_0) \delta \left[ (x-x')^2 \right] \]

Solutions to Maxwell equations are:

\[ A^\mu(x) = A^\mu_{\text{in}}(x) + \frac{\mu_0}{c} \int d^4x' \, G_r(x-x') \, J^\mu(x') \]

If we are given initial incoming field \( A^\mu_{\text{in}} \),

\[ A^\mu(x) = A^\mu_{\text{out}}(x) + \frac{\mu_0}{c} \int d^4x' \, G_a(x-x') \, J^\mu(x') \]

If we have the outgoing field \( A^\mu_{\text{out}} \).

For a point charge moving along \( \vec{r}(t) \):

\[ \rho(\vec{x}, t) = e \, S(\vec{x} - \vec{r}(t)) \quad \vec{J}(\vec{x}, t) = e \, \vec{V}(t) \, S(\vec{x} - \vec{r}(t)) \]

where \( \vec{V}(t) = \frac{d\vec{r}(t)}{dt} \).
Hence, the field of a point charge, in the absence of a medium, is

\[
\Phi(x, t) = \frac{4\pi}{c} \int d^3x' \frac{1}{4\pi} \frac{\Theta(x_0 - x_{0'})}{|x - x'||}.
\]

\[S(x_0 - x_{0'} - x') \in S(x' - \vec{r}(t')) = \]

\[= \int dx_{0'} \frac{\Theta(x_0 - x_{0'})}{|x - x'|} \in S(x_0 - x_{0'} - x') \in S(x - \vec{r}(t')) \]

\[= \int dt' \frac{\mathcal{E}}{|x - \vec{r}(t')|} \in \mathcal{S}(t - t' - \frac{1}{c}|x - \vec{r}(t')|) \]

\[
\Rightarrow t' \text{ has to be determined from the implicit equation: (label } t' = t_{\text{ret}})
\]

\[
t_{\text{ret}} = t - \frac{1}{c} |x - \vec{r}(t_{\text{ret}})|
\]

To integrate, denote \( F(t, t') = t - t' - \frac{1}{c} |x - \vec{r}(t')| \)

\[
\Rightarrow \Phi(x, t) = \int_{-\infty}^{\infty} dt' \frac{\mathcal{E}}{|x - \vec{r}(t')|} \in \mathcal{S}(F(t, t')) =
\]

\[
= \frac{\mathcal{E}}{|x - \vec{r}(t_{\text{ret}})|} \left| \frac{1}{\partial F/\partial t'} \right|_{t' = t_{\text{ret}}}
\]
\[
\frac{\partial F}{\partial t'}\bigg|_{t'=t} = -1 + \frac{1}{c} \frac{(x - \vec{\nu}(t)) \cdot \vec{\nu} / \nu}{|x - \vec{\nu}(t)|}
\]

where \( \vec{\nu}(t) = \frac{d\vec{\nu}(t)}{dt} \)

Defining \( \hat{\nu} = \frac{x - \vec{\nu}(t)}{|x - \vec{\nu}(t)|} \)

and \( \hat{\beta}(t) = \frac{\vec{\nu}(t)}{c} \)

get
\[
\frac{\partial F}{\partial t'}\bigg|_{t'=t} = -1 + \hat{\nu} \cdot \hat{\beta} \Rightarrow \left| \frac{\partial F}{\partial t'}\bigg|_{t'=t} \right| = 1 - \hat{\nu} \cdot \hat{\beta}
\]

as \(|\hat{\beta}| < 1\). \Rightarrow \Phi(x, t) = \left[ \frac{e}{(1 - \hat{\nu} \cdot \hat{\beta}) R} \right]_{\text{rad}}

where \( R^{(r)} = |x - \vec{\nu}(t)| \) and the subscript of "rad" means that we need to evaluate everything at \( t = t_{\text{rad}} \).

Similarly \( \vec{A}(x, t) = \left[ \frac{e \hat{\beta}}{(1 - \hat{\nu} \cdot \hat{\beta}) R} \right]_{\text{rad}} \).

These are Liénard-Wiechert potentials!

To find \( \vec{E} \) and \( \vec{B} \) need to use
\[
\vec{E} = -\nabla \Phi - \frac{i}{c} \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \nabla \times \vec{A}.
\]