Euler-Lagrange equations for $A^\mu$ are

$$\frac{\partial L}{\partial A^\mu} - \partial_\nu \left( \frac{\partial L}{\partial (\partial_\nu A^\mu)} \right) = 0.$$  

$$\frac{\partial L}{\partial A^\mu} = -\frac{1}{c^2} J^\mu,$$

as it is quadratic in $\partial_\nu A^\mu$

$$\frac{\partial L}{\partial (\partial_\nu A^\mu)} = -\frac{1}{16\pi c} 2 (F^\nu{}^\mu - F^\mu{}^\nu) = \frac{1}{4\pi c} F^\nu{}^\mu$$

$$\Rightarrow \frac{\partial_\nu F^\nu{}^\mu}{4\pi c} = -\frac{1}{c^2} J^\mu \Rightarrow \partial_\nu F^\nu{}^\mu = \frac{4\pi c}{c} J^\mu$$

exactly Maxwell equations as we derived.

**Conservation Laws and Energy-Momentum Tensor.**

We have the continuity condition $\partial_\mu J^\mu = 0$ which is an example of a conservation law.

*Noether's theorem* states that for every symmetry there exists a corresponding conservation law.
Imagine a field theory with a Lagrangian $\mathcal{L} = \mathcal{L}(\phi_i, \partial \phi_i)$ that is invariant under coordinate transformations $x^\mu \to x^\mu + \delta x^\mu$.

\[ \frac{\partial \mathcal{L}}{\partial x^\mu} = \frac{\partial \mathcal{L}}{\partial \phi_i} \frac{\partial \phi_i}{\partial x^\mu} + \frac{\partial \mathcal{L}}{\partial (\partial \phi_i)} \frac{\partial (\partial \phi_i)}{\partial x^\mu} = \]

\[ = \partial \mu \left( \frac{\partial \mathcal{L}}{\partial (\partial \phi_i)} \right) \frac{\partial \phi_i}{\partial x^\mu} + \frac{\partial \mathcal{L}}{\partial (\partial \phi_i)} \frac{\partial (\partial \phi_i)}{\partial x^\mu} = \]

\[ = \partial \mu \left[ \frac{\partial \mathcal{L}}{\partial (\partial \phi_i)} \frac{\partial \phi_i}{\partial x^\mu} \right] \Rightarrow \partial \mu \frac{\partial \mathcal{L}}{\partial x^\mu} = \partial \mu (\mathbf{j}_i) \]

\[ = \partial \mu \left[ \frac{\partial \mathcal{L}}{\partial (\partial \phi_i)} \partial \phi_i - \mathbf{j} \cdot \mathbf{L} \right] = 0 \]

\[ \Rightarrow \text{Define Energy-Momentum Tensor} \]

\[ T_{\mu \nu} = \frac{\partial \mathcal{L}}{\partial (\partial \phi_i)} \partial \phi_i - \mathbf{j} \cdot \mathbf{L} \]

or, equivalently,

\[ T_{\mu \nu} = \frac{\partial \mathcal{L}}{\partial (\partial \phi_i)} \partial \phi_i - g_{\mu \nu} \mathbf{L}. \]
As we've just derived the tensor is explicitly conserved: \[ \partial_{\mu} T^{\mu}_{\nu} = 0 \]

Apply these results to EM: \[ \mathcal{L}_{EM} = -\frac{1}{16\pi} F_{\mu\nu}^2 \]

\[ + \mu_{\rho} = \frac{\partial \mathcal{L}}{\partial (\partial_{\nu} A_{\rho})} \partial_{\mu} A_{\rho} - g^{\mu\nu} \mathcal{L}_{EM} \]

\[ \Rightarrow T^{\mu}_{\nu, EM} = \frac{1}{4\pi} F^{\rho\sigma} \partial_{\rho} A_{\sigma} + \frac{1}{16\pi} g^{\mu\nu} F_{\rho\sigma}^2 \]

However, this definition of energy-momentum tensor is not unique, in the sense that one can always add \[ T^{\mu}_{\nu} \rightarrow T^{\mu}_{\nu} + \partial_{\rho} \eta^{\mu\nu} \]
where \[ \eta^{\mu\nu} \] is some anti-symmetric tensor

\[ \Rightarrow \partial_{\mu} T^{\mu}_{\nu} \rightarrow \partial_{\mu} T^{\mu}_{\nu} + \left( \partial_{\rho} \partial_{\nu} \eta^{\mu\nu} = 0 \right) \]

\[ \Rightarrow \text{can use this property to define a symmetric energy-momentum tensor:} \]

\[ T^{\mu}_{\nu} = T^{\nu}_{\mu} \]