$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell}(x^2-1)^\ell = \frac{1}{2^\ell \cdot \ell!} \frac{d^\ell}{dx^\ell} \sum_{m=0}^{\infty} \underbrace{C_\ell^m}_{\frac{\ell!}{m!(\ell-m)!}} (x^2)^m (-1)^{\ell-m}$$

Proof of Rodrigues' formula

$$\frac{d}{dx}\left[(1-x^2) \frac{dP}{dx} \right] + \ell(\ell+1) P = 0$$

$$P_\ell(x) = \frac{(-1)^\ell}{2^\ell \ell!} \frac{d^\ell}{dx^\ell}(1-x^2)^\ell = \frac{(-1)^\ell}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} \sum_{m=0}^{\ell} \frac{\ell!}{m!(\ell-m)!} \underbrace{(-x^2)^m}_{(-1)^m x^{2m}}$$

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_\ell}{dx}\right] = \frac{(-1)^\ell}{2^\ell \ell!} \frac{d}{dx}\left[(1-x^2) \frac{d^{\ell+1}}{dx^{\ell+1}} \sum_{m=0}^{\ell} C_\ell^m (-x^2)^m\right]$$

$$= \frac{(-1)^\ell}{2^\ell \ell!} \frac{d}{dx}\left[(1-x^2) \frac{d^\ell}{dx^\ell} \sum_{m=0}^{\ell} C_\ell^m \, m (-x^2)^{m-1} (-2x)\right]$$

$$= \frac{(-1)^\ell}{2^\ell \ell!} \frac{d}{dx}\left[(1-x^2) \sum_{m=0}^{\ell} C_\ell^m \cdot (2m)(2m-1)\dots(2m-\ell) (-1)^m x^{2m-\ell-1}\right]$$

$$= \frac{(-1)^\ell}{2^\ell \cdot \ell!} \sum_{m=0}^{\ell} C_\ell^m \frac{(2m)!}{(2m-\ell-1)!} (-1)^m \left[(2m-\ell-1)\, x^{2m-\ell-2} - (2m-\ell+1)\, x^{2m-\ell}\right]$$

$m \to m+1$

$$= \frac{(-1)^\ell}{2^\ell \ell!} \sum_{m=0}^{\ell} x^{2m-\ell} (2m-\ell+1) \left[C_\ell^{m+1} \frac{(2m+2)!}{(2m+1-\ell)!} (-1)^{m+1} - C_\ell^m \frac{(2m)!}{(2m-\ell-1)!} (-1)^m \right]$$

$$= \frac{(-1)^{\ell}}{2^{\ell}\ell!}\sum_{m=0}^{\ell} x^{2m-\ell}\,(2m-\ell+1)\Big[\frac{\ell!}{(\ell-m-1)!\,(m+1)!}\,\frac{2(m+1)\,(2m+1)\cdot(2m)!}{(2m+1-\ell)!}(-1)^{m+1} -$$

$$- \frac{\ell!}{m!\,(\ell-m)!}\,\frac{(2m)!}{(2m-\ell-1)!}(-1)^{m}\Big) = \frac{(-1)^{\ell}}{2^{\ell}\ell!}\sum_{m=0}^{\ell} x^{2m-\ell}\Big[\frac{\ell!}{m!\,(\ell-m)!}\,2(\ell-m)\cdot$$

$$\cdot(2m+1)\,\frac{(2m)!}{(2m-\ell)!}(-1)^{m+1} - \frac{\ell!}{m!\,(\ell-m)!}\,\frac{(2m)!}{(2m-\ell)!}\,(2m-\ell+1)(2m-\ell)(-1)^{m}\Big]$$

$$= \frac{(-1)^{\ell}}{2^{\ell}\ell!}\sum_{m=0}^{\infty} C_{\ell}^{m}\,(-1)^{m}\,x^{2m-\ell}\,\frac{(2m)!}{(2m-\ell)!}\Big[-2(\ell-m)(2m+1)-$$

$$- (2m-\ell+1)(2m-\ell)\Big]$$

$$\|$$

$$\Big\{\, 2\ell\ \cancel{4\ell m} + \cancel{4m^2} + \cancel{2m} - \cancel{4m^2} + \cancel{2\ell m} + \cancel{2\ell m}\ \cancel{-2m} - \ell(\ell-1)\Big]$$

$$= -\ell^2 + \ell - 2\ell = -\ell\cdot(\ell+1)$$

$$= -\ell\cdot(\ell+1)\cdot P_{\ell}(x)$$ as desired!