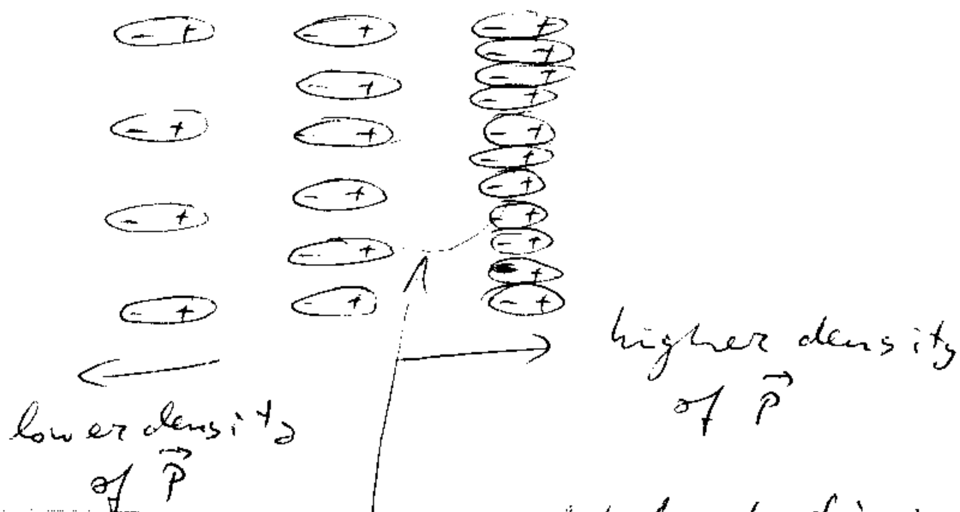
As the bound charge density is $\rho_b = -\vec{\nabla}\cdot\vec{P}$

$\Rightarrow P_{1n} - P_{2n} = -\sigma_b$.

Why is $\rho_b = -\vec{\nabla}\cdot\vec{P}$?

Pictorially:



more likely to find more negative charges in a volume element $\Downarrow$ generates ρ_b!

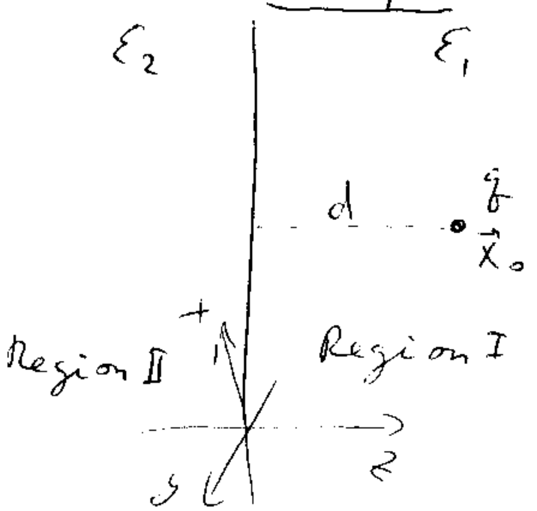
Finally, if $\vec{D} = \varepsilon\vec{E}$

$\Rightarrow \vec{\nabla}\cdot\vec{D} = \varepsilon\vec{\nabla}\cdot\vec{E} = \rho_f \Rightarrow$

$$\Rightarrow \vec{\nabla}\cdot\vec{E} = \frac{\rho_f}{\varepsilon}$$

Boundary-Value Problems with Dielectrics.

Example 1



Region I: $\vec{\nabla}\cdot\vec{D} = \rho_f = \varepsilon_1\vec{\nabla}\cdot\vec{E}$

$$\Rightarrow \vec{\nabla}\cdot\vec{E} = \frac{\rho_f}{\varepsilon_1} = \frac{q}{\varepsilon_1}\delta^3(\vec{x}-\vec{x}_0)$$

$\vec{\nabla}\times\vec{E} = 0 \Rightarrow \vec{E} = -\vec{\nabla}\Phi$.

$\vec{x}_0 = (0, 0, d)$

Region II: $\vec{\nabla} \cdot \vec{E} = 0$, $\vec{\nabla} \times \vec{E} = 0 \Rightarrow \vec{E} = -\vec{\nabla} \Phi$.

Boundary conditions: no surface charge $\Rightarrow$

$\Rightarrow D_{1z} = D_{2z} \Rightarrow \varepsilon_1 E_{1z} = \varepsilon_2 E_{2z}$

$$\left.\begin{aligned} E_{1x} &= E_{2x} \\ E_{1y} &= E_{2y} \end{aligned}\right\} \text{tangential components.}$$

as $\vec{\nabla} \times \vec{E} = 0$ everywhere $\Rightarrow \vec{E} = -\vec{\nabla}\Phi \Rightarrow$ try using image method.

Region I: $\Phi_1(\vec{x}) = \dfrac{1}{4\pi\varepsilon_1} \dfrac{q}{|\vec{x}-\vec{x}_0|} + F(\vec{x})$

where $\nabla^2 F = 0$ in Region I.

Place an image charge q' at $\vec{x}_i = (0,0,-d)$

$$\Rightarrow \Phi_1(\vec{x}) = \frac{1}{4\pi\varepsilon_1} \frac{q}{|\vec{x}-\vec{x}_0|} + \frac{1}{4\pi\varepsilon_1} \frac{q'}{|\vec{x}-\vec{x}_i|}, \quad z > 0$$

Region II: no real charge, try image charge at the position of original charge

$$\Phi_2(\vec{x}) = \frac{1}{4\pi\varepsilon_2} \frac{q''}{|\vec{x}-\vec{x}_0|}, \quad z < 0$$

Matching at the boundary (due to $x \leftrightarrow y$ symmetry, we have 2 equations $\Rightarrow$ 2 equs and 2 unknowns q & q' $\Rightarrow$ OK.)

$$\varepsilon_1 E_{1z} = \varepsilon_2 E_{2z} \Rightarrow \varepsilon_1 \frac{\partial \Phi_1}{\partial z}\Big|_{z=0} = \varepsilon_2 \frac{\partial \Phi_2}{\partial z}\Big|_{z=0}$$

as $$\frac{\partial}{\partial z}\left(\frac{1}{\sqrt{x^2+y^2+(z-d)^2}}\right)\Big|_{z=0} = \frac{+d}{(x^2+y^2+d^2)^{3/2}} =$$

$$= \frac{d}{(\rho^2+d^2)^{3/2}} \Rightarrow \varepsilon_1 \cdot \frac{q}{\varepsilon_1} \frac{d}{(\rho^2+d^2)^{3/2}} + \varepsilon_1 \frac{q'}{\varepsilon_1} \cdot$$

$$\cdot \frac{-d}{(\rho^2+d^2)^{3/2}} = \varepsilon_2 \cdot \frac{q''}{\varepsilon_2} \cdot \frac{d}{(\rho^2+d^2)^{3/2}} \Rightarrow q - q' = q''$$

$$\frac{\partial}{\partial \rho}\left(\frac{1}{\sqrt{\rho^2+(z-d)^2}}\right)\Big|_{z=0} = -\frac{\rho}{(\rho^2+d^2)^{3/2}} \Rightarrow E_{1\rho} = E_{2\rho}$$

$$\Rightarrow \frac{\partial \Phi_1}{\partial \rho}\Big|_{z=0} = \frac{\partial \Phi_2}{\partial \rho}\Big|_{z=0} \Rightarrow \frac{q}{\varepsilon_1} \frac{-\rho}{(\rho^2+d^2)^{3/2}} + \frac{q'}{\varepsilon_1} \cdot$$

$$\cdot \frac{-\rho}{(\rho^2+d^2)^{3/2}} = \frac{q''}{\varepsilon_2} \frac{-\rho}{(\rho^2+d^2)^{3/2}} \Rightarrow \frac{1}{\varepsilon_1}(q+q') = \frac{q''}{\varepsilon_2}$$

$$\Rightarrow q' = -\frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2+\varepsilon_1} q \quad , \quad q'' = \frac{2\varepsilon_2}{\varepsilon_2+\varepsilon_1} q$$

problem solved!

Special cases:

(i) $\varepsilon_1 = \varepsilon_2$ (no boundary) $\Rightarrow q' = 0, \; q'' = q$

(ii) $\varepsilon_1 = \varepsilon_0, \; \varepsilon_2 \to \infty \Rightarrow q' = -q, \; q'' \to 2q$

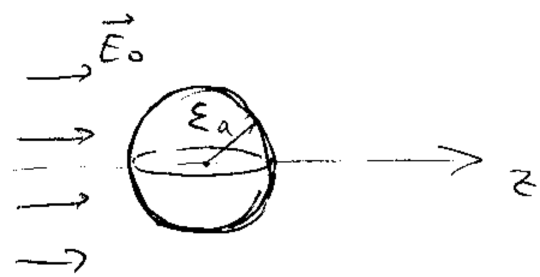
$\Rightarrow$ just like outside of conductor

$\Rightarrow \varepsilon_2 \to \infty \Rightarrow \Phi_2 \to 0 \Rightarrow \vec{E}_2 = 0$

$\vec{E}_1$ is just like outside a conductor.

$\Rightarrow$ Images are not created by surface charges, like it was with conductors. Instead, they are due to jumps in polarization.

Example 2: dielectric sphere in external $\vec{E}$-field.



no free charges $\Rightarrow$

$\vec{\nabla} \cdot \vec{D} = 0$ inside & outside

$\vec{\nabla} \times \vec{E} = 0$ inside & outside

$\vec{D}_{out} = \varepsilon_0 \vec{E}_{out}, \quad \vec{D}_{in} = \varepsilon \vec{E}_{in}$

$\Rightarrow$ as $\vec{\nabla} \times \vec{E} = 0 \Rightarrow \vec{E}_{out} = -\vec{\nabla} \Phi_{out}, \; \vec{E}_{in} = -\vec{\nabla} \Phi_{in}$

$0 = \vec{\nabla} \cdot \vec{D}_{out} = \varepsilon_0 \vec{\nabla} \cdot \vec{E}_{out} = -\varepsilon_0 \nabla^2 \Phi_{out} \Rightarrow \nabla^2 \Phi_{out} = 0$