$$\Rightarrow \alpha_{\ell m} = \frac{4\pi}{2\ell+1}$$ and

$$\boxed{P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_{\ell m}^*(\theta',\varphi') Y_{\ell m}(\theta,\varphi)}$$

addition thm.

Using $\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\gamma)$ we get

$$\boxed{\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell m}^*(\theta',\varphi') Y_{\ell m}(\theta,\varphi)}$$

expansion for $G(\vec{x},\vec{x}')$ in vacuum.

<u>Example</u>: (Dirichlet) Green function outside of conducting sphere: (of radius R) $\Rightarrow$ using method of images

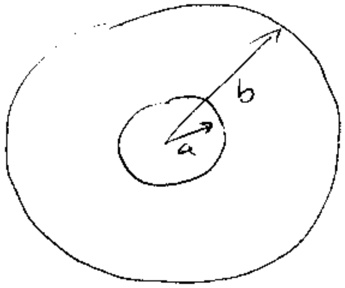
$$G_D(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{R}{r'} \frac{1}{|\vec{x} - \frac{R^2}{r'^2}\vec{x}'|}$$

where $r = |\vec{x}|$, $r' = |\vec{x}'|$. $\Rightarrow$ using the above expansion

$$\Rightarrow G_D(\vec{x},\vec{x}') = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \left[\frac{r_<^\ell}{r_>^{\ell+1}} - \frac{1}{R}\left(\frac{R^2}{rr'}\right)^{\ell+1} \right] \cdot Y_{\ell m}^*(\theta',\varphi') Y_{\ell m}(\theta,\varphi)$$

$$\left(\frac{R^2}{rr'}\right)^{\ell+1}\frac{1}{R} = \frac{R}{r'} \cdot \left(\frac{R^2}{r'}\right)^\ell \cdot \frac{1}{r^{\ell+1}}$$

Another example: find Dirichlet Green function in the region between two concentric spheres of radii a & b:



$$\nabla^2 G_D(\vec{x}, \vec{x}') = -4\pi\, \delta^3(\vec{x} - \vec{x}')$$

$$\delta^3(\vec{x} - \vec{x}') = \frac{1}{r^2}\, \delta(r-r')\, \delta(\varphi-\varphi')\, \delta(\cos\theta - \cos\theta') =$$

$$= \frac{1}{r^2}\, \delta(r-r') \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}^*(\theta', \varphi')\, Y_{\ell m}(\theta, \varphi)$$

$$\Rightarrow \text{look for } G_D(\vec{x}, \vec{x}') = \sum_{\ell, m} g_{\ell m}(r, r')\, Y_{\ell m}^*(\theta', \varphi')\, Y_{\ell m}(\theta, \varphi)$$

$$\Rightarrow \sum_{\ell, m} \left[\frac{1}{r} \frac{\partial^2}{\partial r^2} \left(r\, g_{\ell m}(r, r') \right) - \frac{\ell(\ell+1)}{r^2}\, g_{\ell m}(r, r') \right] Y_{\ell m}^*\, Y_{\ell m} =$$

$$= -\frac{4\pi}{r^2}\, \delta(r-r') \sum_{\ell, m} Y_{\ell m}^*\, Y_{\ell m}$$

$$\Rightarrow \frac{1}{r} \frac{\partial^2}{\partial r^2} \left(r\, g_{\ell m} \right) - \frac{\ell(\ell+1)}{r^2}\, g_{\ell m} = -\frac{4\pi}{r^2}\, \delta(r-r')$$

$$\Rightarrow g_{\ell m}(r, r') = \begin{cases} A_\ell r^\ell + B_\ell r^{-\ell-1}, & r < r' \\ A'_\ell r^\ell + B'_\ell r^{-\ell-1}, & r > r' \end{cases}$$

$g_{\ell m} = 0$ for $r, r' = a, b$ $\Rightarrow$

$$g_{lm}(r,r') = \begin{cases} A_l\left(r^l - a^{2l+1} r^{-l-1}\right), & r<r' \\ B_l'\left(r^{-l-1} - r^l \cdot b^{2l+1}\right), & r>r' \end{cases}$$

$$\Rightarrow g_{lm}(r,r') = C\left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right).$$

Fix the coefficient from $\frac{\partial}{\partial r}(r g_{lm})\Big|_{r=r'-\varepsilon}^{r=r'+\varepsilon} = -\frac{4\pi}{r'}$

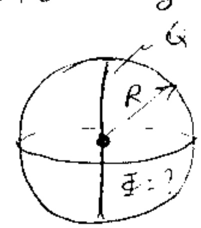
$$\Rightarrow C = \frac{4\pi}{(2l+1)\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)} \quad \Rightarrow \text{finally}$$

$$G_D(\vec{x},\vec{x}') = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{Y_{lm}^*(\theta',\varphi')\,Y_{lm}(\theta,\varphi)}{(2l+1)\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)} \cdot \left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right).$$

take $a\to 0$, $b\to\infty$ $\Rightarrow$ get $\frac{1}{|\vec{x}-\vec{x}'|}$

take $b\to\infty$, a fixed $\Rightarrow$ get the Green function outside a sphere.

<u>Example</u>: Find the field of a uniformly charged stick inside a grounded conducting sphere:



$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int d^3x' \, G_D(\vec{x}-\vec{x}')\rho(\vec{x}') - \int \frac{da'}{4\pi} \underbrace{\Phi(\vec{x}')}_{=0 \text{ here}} \frac{\partial G}{\partial n'}$$

$$\rho(\vec{x}') = \frac{Q}{2b} \frac{1}{2\pi r'^2} \left[\delta(\cos\theta' - 1) + \delta(\cos\theta'+1) \right]$$

$\Rightarrow$ taking $a \to 0$, $b = R$ limit of the obtained Green function we write:

$$\Phi(r,\theta,\varphi) = \frac{1}{\varepsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} Y_{lm}(\theta,\varphi) \cdot \int_0^R dr' \cdot r'^2 \cdot$$

$$\int_{-1}^{1} d\cos\theta' \cdot \underbrace{\int_0^{2\pi} d\varphi' \, Y^*_{lm}(\theta',\varphi')}_{2\pi Y^*_{l0}(\theta',0)\delta_{m0} = 2\pi\sqrt{\frac{2l+1}{4\pi}}\delta_{m0} P_l(\cos\theta')} r_<^l \left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{R^{2l+1}} \right) \cdot$$

$$\cdot \frac{Q}{2R} \frac{1}{2\pi r'^2} \left[\delta(\cos\theta'-1) + \delta(\cos\theta'+1) \right] =$$

$$= \frac{1}{\varepsilon_0} \sum_{l=0}^{\infty} \frac{1}{2l+1} \cdot 2\pi \sqrt{\frac{2l+1}{4\pi}} \, Y_{l0}(\theta,\varphi) \cdot \int_0^R dr' \int_{-1}^{1} d\cos\theta' \cdot$$

$$\cdot P_l(\cos\theta') \left[\delta(\cos\theta'-1) + \delta(\cos\theta'+1) \right] \frac{Q}{4\pi R} \cdot \left(\frac{r_<^l}{r_>^{l+1}} - \frac{(rr')^l}{R^{2l+1}} \right)$$

$$= \frac{Q}{2R\varepsilon_0} \frac{1}{4\pi} \sum_{l=0}^{\infty} P_l(\cos\theta) \cdot \left[\overbrace{P_l(1)}^{=1} + \underbrace{P_l(-1)}_{(-1)^l} \right] \int_0^R dr' \cdot$$

$$\cdot \left(\frac{r_<^l}{r_>^{l+1}} - \frac{(rr')^l}{R^{2l+1}} \right)$$

$\Rightarrow$ only even $l = 2j$ contribute.

Performing r'-integral:

$$\int_0^R dr' \left(\frac{r_<^\ell}{r_>^{\ell+1}} - \frac{(rr')^\ell}{R^{2\ell+1}}\right) = \int_0^r dr' \left(\frac{r'^\ell}{r^{\ell+1}}\right) + \int_r^R dr' \frac{r^\ell}{r'^{\ell+1}} -$$

$$- \frac{1}{\ell+1}\frac{R^{\ell+1}}{R^{2\ell+1}} \cdot r^\ell = \frac{1}{\ell+1} + r^\ell \left(\frac{-1}{\ell}\right)(r')^{-\ell}\Big|_r^R - \frac{1}{\ell+1}\frac{r^\ell}{R^\ell} =$$

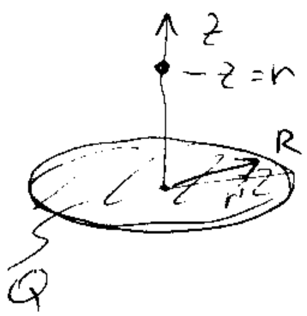
$$= \frac{1}{\ell+1}\left(1-\frac{r^\ell}{R^\ell}\right) + \frac{1}{\ell}\left(1-\frac{r^\ell}{R^\ell}\right) = \frac{2\ell+1}{\ell(\ell+1)}\left(1-\frac{r^\ell}{R^\ell}\right)$$

$$\Rightarrow \boxed{\Phi(r,\theta) = \frac{Q}{8\pi R\varepsilon_0}\sum_{j=0}^{\infty} \frac{4j+1}{j(2j+1)}\left(1-\frac{r^{2j}}{R^{2j}}\right)P_{2j}(\cos\theta)}$$

where, for $\ell = 0$ ($j=0$) the coefficient becomes

$2\ln R/r$ (just take $j \to 0$ limit of it)

<u>Example:</u> uniformly charged disk of radius R



find $\Phi(r,\theta,\varphi)$.

Remember: we need potential along z-axis as a series in r:

$$\Phi = \frac{1}{4\pi\varepsilon_0}\frac{Q}{\pi R^2}\int_0^{2\pi} d\varphi \int_0^R dr' \cdot r' \frac{1}{\sqrt{r'^2+z^2}} = \frac{Q}{4\pi\varepsilon_0}\frac{2}{R^2}\cdot\sqrt{r'^2+z^2}\Big|_0^R =$$

$$= \frac{Q}{2\pi\varepsilon_0 R^2}\left(\sqrt{R^2+z^2} - z\right) \Rightarrow$$ let's look at large distances:

$$\Phi = \frac{Q\,z}{2\pi\varepsilon_0 R^2}\left(\sqrt{1+R^2/z^2}-1\right) \approx \frac{Q\,z}{2\pi\varepsilon_0 R^2}\left(\frac{1}{2}\frac{R^2}{z^2} - \frac{1}{8}\frac{R^4}{z^4} + \dots\right) \Rightarrow \text{put } z = r$$

$$\Rightarrow \Phi(z=r) = \frac{Q}{4\pi\varepsilon_0 R^2}\left[\frac{R^2}{r} - \frac{1}{4}\frac{R^4}{r^3} + \dots\right]$$

compare with $\Phi \sim \sum_\ell \left(A_\ell r^\ell + B_\ell r^{-\ell-1}\right) P_\ell(\cos\theta)$

to include Legendre polynomials

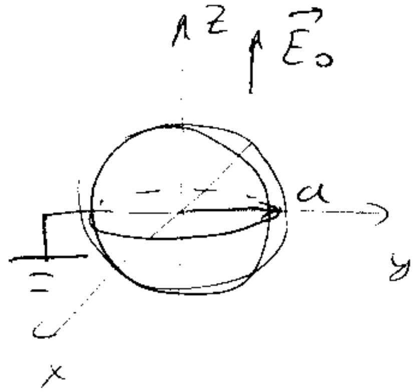
$$\Phi(r,\theta) = \frac{Q}{4\pi\varepsilon_0 R^2}\left[\frac{R^2}{r} P_0(\cos\theta) - \frac{1}{4}\frac{R^4}{r^3} P_2(\cos\theta) + \dots\right]$$

$\Rightarrow$ as $P_0(x) = 1 \Rightarrow$

$$\Phi(r,\theta) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r}\left[1 - \frac{R^2}{4r^2} P_2(\cos\theta)\right]$$

can explicitly see corrections to point charge approximation at larger r.

Another example of Legendre polynomial expansion.



sphere (grounded & conducting) in uniform electric field:

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right) P_\ell(\cos\theta)$$

at $r \to \infty$ have only potential due to $\vec{E}_0$ =>

$$\Rightarrow \quad \Phi(r\to\infty) = -E_0 z = -E_0 r\cos\theta = -E_0 r P_1(\cos\theta)$$

$$\Rightarrow \quad A_1 = -E_0, \quad A_\ell = 0 \quad \text{if } \ell \neq 1.$$

$$\Rightarrow \quad \Phi(r,\theta) = \sum_{\ell=0}^{\infty} B_\ell r^{-\ell-1} P_\ell(\cos\theta) - E_0 r P_1(\cos\theta)$$

at $r=a$: $\Phi(a,\theta) = -E_0 a P_1(\cos\theta) + \sum_{\ell=0}^{\infty} B_\ell \cdot$

$\cdot\, a^{-\ell-1} P_\ell(\cos\theta) = 0$ => due to orthogonality &

& completeness of P_ℓ's : $B_\ell = 0$ if $\ell \neq 1$

$$B_1 = E_0 a^{\overset{=1}{\ell}+2} = E_0 a^3.$$

$$\Rightarrow \quad \Phi(r,\theta) = -E_0 r P_1(\cos\theta)\left(1 - \frac{a^3}{r^3}\right)$$

cf. with our earlier take on this problem.