Midterm Review

Coulomb's Law: $F = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^2}$

$$\Rightarrow \quad \vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int d^3x' \, \frac{\rho(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|^3} (\vec{x} - \vec{x}\,')$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int d^3x' \, \frac{\rho(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|}$$

δ - functions: (i) $\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}$ (ii) $\int_{-\infty}^{\infty} dx \, f(x)\,\delta(x) = f(0)$

$$\boxed{\nabla^2 \frac{1}{|\vec{x} - \vec{x}\,'|} = -4\pi \, \delta^3(\vec{x} - \vec{x}\,')}$$

$\vec{\nabla} \cdot \vec{E} = \rho/\varepsilon_0$ Gauss's Law

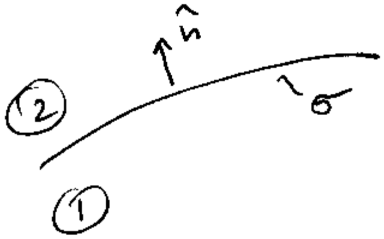
$\oint_S \vec{E} \cdot \hat{n} \, da = \frac{1}{\varepsilon_0} \int_V d^3x \, \rho(\vec{x}) = \frac{Q}{\varepsilon_0}$ integral form

$$\vec{\nabla} \times \vec{E} = 0 \quad \Rightarrow \quad \vec{E} = -\vec{\nabla}\Phi \quad \Rightarrow$$

$\boxed{\nabla^2 \Phi = -\rho/\varepsilon_0}$ Poisson eqn.

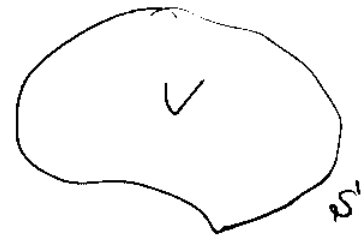
$\boxed{\nabla^2 \Phi = 0}$ Laplace eqn. (if $\rho = 0$)

boundaries: $\boxed{\left(\vec{E}_2 - \vec{E}_1\right) \cdot \hat{n} = \frac{1}{\varepsilon_0}\sigma}$ Gauss's Law



$\boxed{E_{2t} = E_{1t}}$ $(\vec{\nabla} \times \vec{E} = 0)$ & Stokes's th'm

Dirichlet b.c. problem:



Φ given on S

solution
$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3x'\, G_D(\vec{x},\vec{x}\,')\rho(\vec{x}\,') - \frac{1}{4\pi}\oint_S \Phi(\vec{x}\,')\frac{\partial G_D(\vec{x},\vec{x}\,')}{\partial n'}\,da'$$

where $G_D(\vec{x},\vec{x}\,')$ is Dirichlet Green ftn:

$$\nabla'^2 G_D(\vec{x},\vec{x}\,') = -4\pi\,\delta^3(\vec{x}-\vec{x}\,')$$

$$G_D(\vec{x},\vec{x}\,') = 0 \quad \text{for } \vec{x}\,' \in S.$$

Neumann Problem: $\frac{\partial \Phi}{\partial n}$ is given on S.

solution
$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3x'\, G_N(\vec{x},\vec{x}\,')\rho(\vec{x}\,') + \frac{1}{4\pi}\oint_S da'\, G_N(\vec{x},\vec{x}\,')\frac{\partial \Phi}{\partial n'} + \langle \Phi \rangle_{\text{surface}}$$

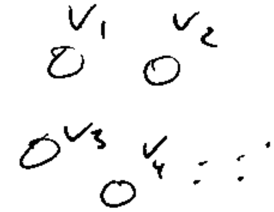
$$\nabla'^2 G_N(\vec{x},\vec{x}\,') = -4\pi\,\delta^3(\vec{x}-\vec{x}\,'), \quad \frac{\partial G_N}{\partial n'} = \frac{-4\pi}{S} \text{ for } \vec{x}\,' \in S.$$

Electrostatic Energy

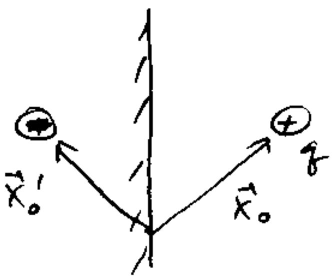
$$W = \frac{1}{2}\int d^3x\, \rho(\vec{x})\, \Phi(\vec{x}) = \frac{\varepsilon_0}{2}\int d^3x\, |\vec{E}|^2(\vec{x})$$

capacitance

$$Q_i = \sum_{j=1}^{n} C_{ij} V_j .$$

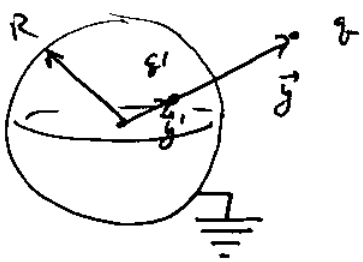


Method of Images



$$\Phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|\vec{x}-\vec{x}_0|} - \frac{1}{|\vec{x}-\vec{x}_0'|}\right]$$

used to satisfy b.c.



$$q' = -q\frac{R}{y}, \quad y' = \frac{R^2}{y}$$

$$\Phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|\vec{x}-\vec{y}|} - \frac{R}{y}\frac{1}{\left|\vec{x} - \frac{R^2}{y^2}\vec{y}\right|}\right]$$

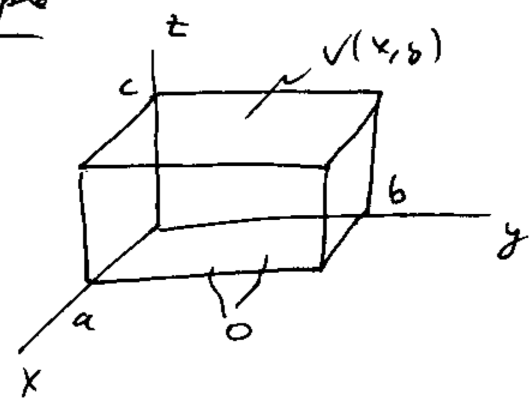
Separation of Variables

① Rectangular Coordinates: $\Phi(x,y,z) = X(x)\,Y(y)\,Z(z)$

$\nabla^2 \Phi = 0 \Rightarrow$

$$\begin{cases} X(x) = e^{\pm i\alpha x} \\ Y(y) = e^{\pm i\beta y} \\ Z(z) = e^{\pm \gamma z}, \quad \gamma^2 = \alpha^2 + \beta^2 \end{cases}$$

Example



$$\Phi(x,y,z) = \sum_{n,m=1}^{\infty} A_{nm} \sin\left(\frac{\pi n}{a} x\right) \cdot \sin\left(\frac{\pi m}{b} y\right) \sinh\Big(z \underbrace{\sqrt{\left(\frac{\pi n}{a}\right)^2 + \left(\frac{\pi m}{b}\right)^2}}_{=\gamma_{nm}}\Big)$$

where $$A_{nm} = \frac{4}{ab\sinh(\gamma_{nm} c)} \int_0^a dx \int_0^b dy\, V(x,y) \sin\left(\frac{\pi n}{a} x\right) \sin\left(\frac{\pi m}{b} y\right).$$

② Cylindrical Coordinates:

A. z - indep. case $\quad \Phi(\rho,\varphi) = R(\rho)\,\Psi(\varphi)$

$$\begin{cases} \Psi(\varphi) = e^{\pm i\nu\varphi} & (\nu \neq 0) \quad R(\rho) \sim a + b \ln\rho \quad \text{if } \nu = 0 \\ R(\rho) = a\rho^{\nu} + b\rho^{-\nu} \end{cases}$$

$\Rightarrow$ solution is
$$\Phi(\rho,\varphi) = a_0 + b_0 \ln\rho + \sum_{n=1}^{\infty} \left[a_n \rho^n \sin(n\varphi + \alpha_n) + b_n \rho^{-n} \sin(n\varphi + \beta_n) \right]$$

B. z - dep. case $\quad \Phi(\rho,\varphi,z) = R(\rho)\,Q(\varphi)\,Z(z)$

$$\begin{cases} Z(z) = e^{\pm k z} \\ Q(\varphi) = e^{\pm i\nu\varphi} \\ R(\rho) = J_\nu(k\rho) \text{ or } N_\nu(k\rho) \end{cases}$$ Bessel ftns of the 1st & 2nd kind

or

$$\begin{cases} Z(z) = e^{\pm i k z}, \quad Q(\varphi) = e^{\pm i\nu\varphi} \\ R(\rho) = I_\nu(k\rho) \text{ or } K_\nu(k\rho) \end{cases}$$ modified Bessel ftns

$\overline{\underline{V}}$

$$\int_0^a d\rho \cdot \rho \cdot J_\nu\left(x_{\nu n}\frac{\rho}{a}\right) J_\nu\left(x_{\nu n'}\frac{\rho}{a}\right) = \frac{a^2}{2}\delta_{nn'}\left[J_{\nu+1}(x_{\nu n})\right]^2$$

where $J_\nu(x_{\nu n}) = 0$, $n = 1, 2, \ldots$ roots of Bessel ftn.

(3) Spherical coordinates: $\Phi(r, \theta, \varphi) = \frac{u(r)}{r} P(\theta) Q(\varphi)$

$\Rightarrow Q(\varphi) = e^{\pm i m\varphi}$, $\frac{u(r)}{r} = A_{\ell m} r^\ell + B_{\ell m} r^{-\ell-1}$

A. Azimuthally symmetric case: $m = 0$, $Q = \text{const.}$

$P(\cos\theta) = P_\ell(\cos\theta)$, $\ell = 0, 1, 2, \ldots$

$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell$ Rodrigues formula.

$$\int_{-1}^{1} dx \, P_\ell(x) \, P_{\ell'}(x) = \frac{2}{2\ell+1}\delta_{\ell\ell'}$$

solution of Laplace eqn:

$$\Phi(r, \theta) = \sum_{\ell=0}^{\infty}\left[A_\ell r^\ell + B_\ell r^{-\ell-1}\right] P_\ell(\cos\theta)$$

$$\frac{1}{|\vec{x} - \vec{x}'|} = \sum_{\ell=0}^{\infty}\frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\gamma)$$, γ = angle between $\vec{x}$ & $\vec{x}'$

$r_{\gtrless} = {}^{\max}_{\min}\{r, r'\}$.